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WEIGHT FUNCTIONS AND FUNDAMENTAL FIELDS FOR THE PENNY-SHAPED AND THE HALF-PLANE CRACK IN THREE-SPACE

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Abstract—Weight functions permit to represent stress intensity factors as weighted averages of the externally impressed boundary tractions and body forces. In this paper the relevant weight functions of all three modes are represented in closed form with the aid of elementary transcendentals. Their derivation as displacements of fundamental fields is simple and without recourse to integral transforms of any kind.

1. INTRODUCTION

We shall be concerned with the calculation of stress intensity factors within the realm of the classical theory of linear elasticity of isotropic bodies. Poisson's parameter ν and the shear modulus μ will be used to describe the elastic material. In the most general situation one considers an elastic body V with a crack C, such as shown in Fig. 1. Let V be subjected to distributed body forces of density **F** and let the surface S of V (the faces of C included) be under distributed tractions **T**. **F** and **T** are vectors, forming fields throughout V and along S respectively. Let the system of these fields be self-equilibrated. The body V responds to them with a field of displacements, strains and stresses. Along the edge of the crack the stress field is generally unbounded and characterized by three stress intensity factors $k_1(Q)$, $k_2(Q)$ and $k_3(Q)$ at a generic point Q of the edge. (The frequently used "engineering" intensity factors $k_{\rm I}$, $k_{\rm II}$ and $k_{\rm III}$ are obtained by multiplying the k_j by $\sqrt{\pi}$.) For a fixed Q the factor $k_j(Q)$ is a linear functional of the fields of **F** and **T**; mindful of the representation of linear functionals in certain function spaces one would expect a representation

$$k_j(Q) = \int_V (\mathbf{F}, W^*)\,\mathrm{d}V + \int_S (\mathbf{T}, W^{*\prime})\,\mathrm{d}S \tag{1.1}$$

where W^*, $W^{*\prime}$ stand for vector fields in V and on S respectively. The symbol (**A**, **B**) denotes the scalar product of vectors **A** and **B**. W^* and $W^{*\prime}$ depend on the field location P, on Q as a parameter and on the choice of j. They do not depend on **F** and **T**. We call them weight functions and write

$$W^* = W_j^*(P, Q), \qquad W^{*\prime} = W_j^{*\prime}(P, Q) \tag{1.2}$$

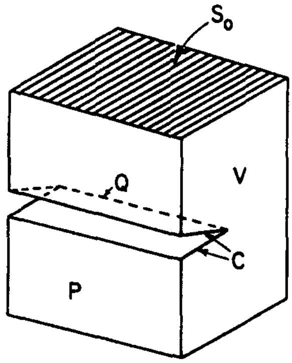


Fig. 1.

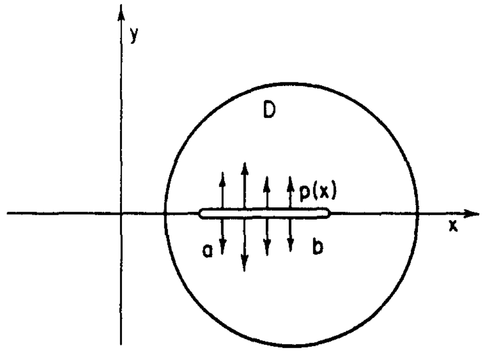


Fig. 2.

more explicitly. Once available the weight functions can be used to calculate the stress intensity factors for all possible load systems **F**, **T**.

Formula (1.1) reduces to a much simpler form for plane strain deformation. Figure 2 shows an elastic region D in the Cartesian (x, y) plane with a crack along the interval (a, b) of the x-axis. Let D be subject to tractions only, and in particular to a generic pressure distribution $p(x)$ along the crack faces. In this case the stress intensity factor k_1 at the crack tip b can be represented by

$$k_1 = \frac{\sqrt{2}}{\pi} \int_a^b p(x) m(x) \, \mathrm{d}x \tag{1.3}$$

where the "crack face weight function" $m(x)$ satisfies the normalization condition

$$\lim_{x \to b-0} (b-x)^{1/2} \cdot m(x) = 1. \tag{1.4}$$

In the case of the Griffith crack (D is the full plane with the crack as shown in Fig. 2) Muskhelishvili's theory permits to determine the field responding to $p(x)$ in closed form with the aid of Cauchy integrals. Retrieving k_1 from that field one finds

$$m(x) = \sqrt{\frac{x-a}{(b-a)(b-x)}}. \tag{1.5}$$

Formula (1.3) can be generalized so as to yield k_1 and k_2 under more general conditions of loading.

It is trivial to interpret weight functions as influence functions. As an example we derive from (1.3) that

$$\frac{\sqrt{2}}{\pi} m(t) = k_1 \qquad \text{for} \qquad p(x) = \delta(x-t). \tag{1.6}$$

Here δ stands for Dirac's function and describes a concentrated load at location $x = t$. The interpretation (1.6) makes $m(x)$ an abstract from infinitely many fields. For each t of the interval (a, b) there is a field responding to $\delta(x-t)$; this field yields a k_1 and thus the particular value of $m(x)$ at $x = t$. Fortunately it is not necessary to procure weight functions in this manner. More powerful methods are available.

Let us label as "general" any method by which we determine the k_j in two steps: (1) the field of displacements and stresses in V in response to **F**, **T** is calculated; (2) from this field we retrieve the k_j for any wanted edge point Q. If the field in V can be found for generic distributions of **F** and **T** then formulas of the type (1.1) follow suit. The same can be achieved if a general method permits to analyse the response to concentrated loads, provided that the points of load application are generically prescribed. If we are content with information on stress intensity factors only, then a general method provides more than we want. It is conceivable that there are methods of crack analysis which yield stress intensity factors, but nothing more. We shall denote them as "special". It is natural to expect the effort to implement a special method to be much smaller than the effort for a general one.

The literature of stress intensity factors is based to a considerable extent on general methods, applied to distributed or concentrated loads. We mention [1, 5, 6, 8, 16–18] in particular. The penny-shaped crack is no exception. Apart from [4] most of its extensive literature is about general methods in one form or another. The crack can be considered as a limit case of oblate spheroidal voids. Such voids are analysed in Neuber's book[9] with the aid of oblate spheroidal coordinates. Applying the latter and Neuber's general formulae Sack[15] determined the energy of deformation for a crack opened by uniform pressure. At about the same time Sneddon[17] found, among other results,

$$k_1 = \frac{2}{\pi} p_0 \sqrt{a} \tag{1.7}$$

for a crack of radius a, opened by the pressure p_0. His analysis employs Hankel transforms and dual integral equations. Following his example Barenblatt[1] applied Hankel transforms in order to deal with an axisymmetric pressure distribution under the crack faces. Let cylindrical coordinates r, θ and z be associated with rectangular Cartesian coordinates x, y and z through $x = r\cos\theta$, $y = r\sin\theta$. For a crack defined by $z = 0$, $r < a$ (Fig. 3) and subject to a pressure distribution $p(r)$ Barenblatt finds

$$k_1 = \frac{2}{\pi}\int_0^a \frac{p(r) r \, \mathrm{d}r}{\sqrt{a(a^2 - r^2)}}. \tag{1.8}$$

The furthest reaching result obtained by a general method[5, 18] seems to be

$$\begin{gathered} k_1(\theta') = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^a M(r,\theta,\theta') p(r,\theta) r \, \mathrm{d}r \, \mathrm{d}\theta \qquad \text{with} \\ M = \sqrt{a^2 - r^2}/\sqrt{a}\cdot d^2;\; d^2 = a^2 + r^2 - 2ar\cos(\theta - \theta') \end{gathered} \tag{1.9}$$

for a crack under a general pressure distribution $p(r,\theta)$; note that k_1 depends on the angle θ' of the edge point.

Many other cases of loading appear in [6, 16, 18]. None of them matches the generality of (1.9). A comprehensive description of W^*, $W^{*\prime}$ in (1.1) for the case of the penny-shaped crack and all three modes of deformation is still missing.

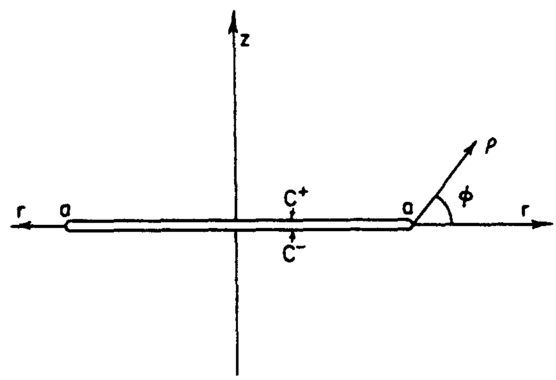


Fig. 3.

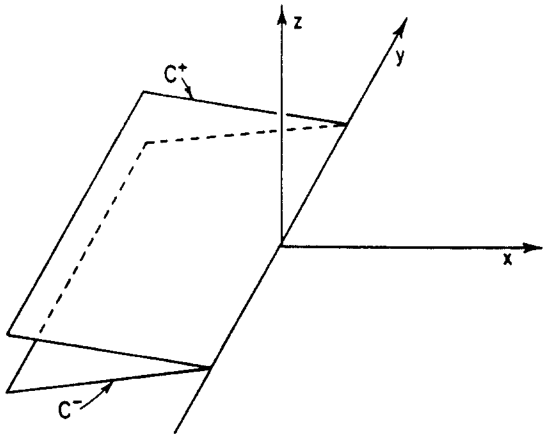


Fig. 4.

About the same can be said for the literature on the half-plane crack (Fig. 4). With the exception of [4, 12] the emphasis has been on general methods. Special concentrated loads are considered in [6] and dealt with the aid of functional transforms, in particular Kantorovich–Lebedev transforms for modes II and III. A recent study of crack faces under concentrated loads is given in [8]. The analysis in [12] covers the case of body forces and provides information for the weight functions of mode I in explicit form. (The main subject of [12] is the perturbation of crack front position; here we refer to it only in the context of half-plane crack analysis.)

In the sequel, a special method will lead us to the relevant weight functions of all modes for both the penny-shaped and the half-plane crack. No recourse to integral transforms will be taken and nothing more complicated than harmonic functions, elementary-transcendental in their variables will be met. All weight functions will be given in closed form.

The special method is not new. It makes use of a particular property of weight functions. Introduced, discussed and applied in [2–4, 7, 10–14], the property identifies the vector fields W^*, $W^{*\prime}$ in (1.1) as the displacements of *one* field of elastic deformation, also known as fundamental field. Here is the place to refer to [12] again. The paper's method is special and uses features of weight function theory. A fundamental field displays neither body forces nor surface tractions. It has unbounded displacements and infinite strain energy. This admits its existence outside the family of those fields to which the uniqueness theorem of elasticity applies. The fundamental fields of a given elastic body with one or more cracks form a linear manifold. They stay fundamental upon multiplication of all field quantities by the same scalar; the sum of two fields is also fundamental. To achieve our goal it will suffice to find and describe certain relevant fundamental fields.

2. REGULAR FIELDS. PRELIMINARIES OF FUNDAMENTAL FIELDS

We begin with the commonly known asymptotic relations for displacements and stresses near an edge point of the crack. It is convenient to list them for the configuration of the half-plane crack (Figs 4 and 5). In the frame of a rectangular Cartesian coordinate system with axes x, y and z, the crack occupies the left half of the (x, y) plane, $x < 0$. The crack faces C^+ and C^- can be distinguished by $\varphi = \pi, -\pi$ respectively where ρ and φ represent polar coordinates in the (x, z) plane. We introduce

$$\alpha = \sqrt{2\rho}\cdot\cos\varphi/2, \qquad \beta = \sqrt{2\rho}\cdot\sin\varphi/2 \tag{2.1}$$

$$\gamma_C = \cos\varphi/2\cos 3\varphi/2, \qquad \gamma_S = \sin\varphi/2\sin 3\varphi/2. \tag{2.1a}$$

Let u, v and w denote elastic displacements in the direction of x, y and z respectively and let σ_x, τ_{xy}, etc. denote the stresses in the familiar manner. The asymptotic relations for u, v and w are given by

$$\begin{aligned} 2\mu(u-u_0) &\sim k_1\alpha[2(1-\nu)-\cos^2\varphi/2]+k_2\beta[2(1-\nu)+\cos^2\varphi/2] \\ 2\mu(w-w_0) &\sim k_1\beta[2(1-\nu)-\cos^2\varphi/2]-k_2\alpha[2(1-\nu)-2+\cos^2\varphi/2] \\ \mu(v-v_0) &\sim -k_3\beta. \end{aligned} \tag{2.2}$$

Here the stress intensity factors k_1, k_2 and k_3 shall be smooth functions of y; so shall the (suitably chosen) terms u_0, v_0 and w_0. Strains follow suit, and so do the stresses. The latter are governed by

$$\begin{aligned} &\sigma_x \sim [k_1\alpha(1-\gamma_S)-k_2\beta(2+\gamma_C)]/2\rho \\ &\sigma_z \sim [k_1\alpha(1+\gamma_S)+k_2\beta\gamma_C]/2\rho\,;\ \sigma_y \sim \nu(\sigma_x+\sigma_z) \\ &\tau_{xz} \sim [k_1\beta\gamma_C+k_2\alpha(1-\gamma_S)]/2\rho \\ &\tau_{yz} \sim -k_3\alpha/2\rho, \qquad \tau_{yx} \sim k_3\beta/2\rho. \end{aligned} \tag{2.3}$$

All preceding relations have the general form

$$\omega \sim \rho^\lambda f(y,\varphi); \tag{2.4}$$

it applies to the approach $\rho \to 0$ and shall mean that

$$\omega = \rho^\lambda f(y,\varphi)+o(\rho^\lambda). \tag{2.5}$$

Let $y_1 < y < y_2$ be some segment of the edge of the crack. A field of displacements, strains and stresses in the structure of the half-plane crack will be called regular along that segment if the relations (2.2) and (2.3) hold with suitable u_0, v_0, w_0; k_1, k_2, k_3 along that segment. We call the field regular if the segment coincides with the whole edge. These definitions can be extended to other elastic structures.

For our purposes it will be preferable to replace some of the standard formulas in (2.2) and (2.3) by equivalents of the form

$$2\mu(u-u_0) \sim -U_x+4(1-\nu)A\,;\ 2\mu(w-w_0) \sim -U_z+4(1-\nu)B \tag{2.6}$$

$$\sigma_x \sim U_{zz}, \qquad \sigma_z \sim U_{xx}, \qquad \tau_{xz} \sim -U_{xz}. \tag{2.7}$$

Here and in the sequel we use coordinate denotations as subscripts in order to indicate partial derivatives; stress denotations σ_x, τ_{xy}, etc. are exempt. In (2.6) A and B are related to one another by

$$A_x = B_z, \qquad A_z = -B_x \tag{2.8}$$

and to U by

$$U_{xx}+U_{zz} = 4A_x. \tag{2.9}$$

If one replaces "$\sim$" by "$=$" in (2.6) and (2.7) one obtains Muskhelishvili's general representation of plane strain fields without body forces; "plane strain" refers to the (x,z) plane. U is the Airy stress function, $A+iB$ is analytic in $x+iz$ and identical with Muskhelishvili's φ. In (2.6) and (2.7) A, B and U are given by

$$\begin{gathered} 2A = k_1\alpha+k_2\beta\,;\ 2B = k_1\beta-k_2\alpha \\ U = k_1U_1+k_2U_2 \qquad \text{with} \qquad 3U_1 = \alpha^3, \qquad U_2 = -\alpha^2\beta. \end{gathered} \tag{2.10}$$

The functions α and β play an important role here and later. The derivations in (2.6) and (2.7) are easily carried out if we observe

$$\rho_x = \rho\varphi_z = \cos\varphi\,;\ \rho_z = -\rho\varphi_x = \sin\varphi. \tag{2.11}$$

For the convenience of the reader we also list

$$\begin{gathered}\alpha^2 = \rho(1+\cos\varphi) = \rho+x\,;\ \beta^2 = \rho(1-\cos\varphi) = \rho-x\\ \alpha\beta = \rho\sin\varphi = z\end{gathered} \tag{2.12}$$

$$\alpha_x = \beta_z = \alpha/2\rho\,;\ \alpha_z = -\beta_x = \beta/2\rho \tag{2.12a}$$

$$\begin{aligned}2U_{1x} &= (1+\cos\varphi)\alpha, &\quad 2U_{1z} &= (1+\cos\varphi)\beta\\ U_{2x} &= -U_{1z}, & U_{2z} &= U_{1x}-2\alpha,\end{aligned} \tag{2.13}$$

and, for later use,

$$\begin{gathered}4\rho U_{1xx} = (1+\cos\varphi+2\sin^2\varphi)\alpha\,;\ 4\rho U_{1xz} = (-1-\cos\varphi+2\sin^2\varphi)\beta\\ U_{2xx} = -U_{1xz}, \qquad U_{2xz} = U_{1xx}-\alpha/\rho.\end{gathered} \tag{2.13a}$$

Now we introduce a different type of field whose quantities we mark by an asterisk. We set first

$$\begin{aligned}\alpha^* &= \alpha_x, &\quad \beta^* &= \beta_x\,;\\ U_1^* &= U_{1x}, & U_2^* &= U_{2x}\end{aligned} \tag{2.14}$$

and define

$$U^* = m_1U_1^*+m_2U_2^*, \qquad 2A^* = m_1\alpha^*+m_2\beta^*, \qquad 2B^* = m_1\beta^*-m_2\alpha^*. \tag{2.14a}$$

The displacements of the field shall satisfy the asymptotic relations

$$\begin{gathered}2\mu u^* \sim -U_x^*+4(1-\nu)A^*, \qquad 2\mu w^* \sim -U_z^*+4(1-\nu)B^*\\ \mu v^* \sim -m_3\beta^*\end{gathered} \tag{2.15}$$

m_1, m_2, m_3 are coefficients, independent of x and z, but permitted to depend on y; they bear no relation to the function $m(x)$ in (1.3)–(1.6). We assume that m_1, m_2 and m_3 are smoothly defined along the whole edge and that at least one of them does not vanish identically. The stresses of the field are implied by (2.15). We write

$$\begin{gathered}\sigma_x^* \sim U_{zz}^*, \qquad \sigma_z^* \sim U_{xx}^*, \qquad \tau_{xz}^* \sim -U_{xz}^*\\ \sigma_y^* \sim \nu(\sigma_x^*+\sigma_z^*), \qquad \tau_{yz}^* \sim -m_3\beta_z^*, \qquad \tau_{yx}^* \sim -m_3\beta_x^*.\end{gathered} \tag{2.16}$$

If y' denotes an edge point where one of the m_k does not vanish then the field has infinite energy of deformation in any neighborhood of y'. A field in the structure of the half-plane crack will be called *ordinary* fundamental if the following conditions are met: (a) for $\rho\to 0$ the asymptotic relations (2.15) and (2.16) hold along the edge with suitably chosen coefficients m_k in accord with the assumptions above; (b) there are no body forces; no tractions appear on C^+, C^-; (c) the stresses go to zero as $\rho\to\infty$. Extending the well-known mode distinction of regular fields we shall speak of an ordinary fundamental field of mode I if $m_1 \neq 0$, $m_2 = 0$, $m_3 = 0$. In similar vein we define modes II and III.

Ordinary fundamental fields form a linear manifold. It is not large enough to provide all cases of important weight functions. Certain limits of ordinary fundamental fields have to be adjoined. The mode distinction will follow suit.

Analogously fundamental fields for other cracked structures can be defined. Before we do this for the case of the penny-shaped crack we have to consider an important line integral involving both a regular and an ordinary fundamental field. We exclude from the structure of the half-plane crack the interior of a cylinder ρ = constant. The remaining elastic body displays certain tractions on the surface ρ = constant; one set of tractions with components X, Y and Z is due to a regular field with displacements u, v and w; another set with components X^*, Y^* and Z^* comes from an ordinary fundamental field with displacements u^*, v^* and w^*. Let $\mathscr{L}$ be the intersection of the cylinder with a plane y = constant, as shown in Fig. 5. Along $\mathscr{L}$ we take the integral

$$I(\rho, y) = \int_{-\pi}^{\pi} (uX^* + vY^* + wZ^* - u^*X - v^*Y - w^*Z)\rho \, \mathrm{d}\varphi. \tag{2.17}$$

At least in an asymptotic sense the tractions X^*, Y^* and Z^* on $\mathscr{L}$ have no force resultant; more precisely

$$I_0(\rho, y) = \int_{\mathscr{L}} (u_0X^* + v_0Y^* + w_0Z^*)\rho \cdot \mathrm{d}\varphi \to 0 \quad \text{as} \quad \rho \to 0 \tag{2.18}$$

for any set of constants u_0, v_0, w_0. To the difference

$$I(\rho, y) - I_0(\rho, y) = \int_{-\pi}^{\pi} [(u-u_0)X^* + (v-v_0)Y^* + (w-w_0)Z^* - u^*X - v^*Y - w^*Z]\rho \, \mathrm{d}\varphi$$

one can apply all the asymptotic relations for the regular and the ordinary fundamental field. It turns out that the products ρXu^*, $\rho X^*(u-u_0)$, etc. are asymptotically independent of ρ; their leading terms are functions of φ alone. Consequently $I - I_0$ has a finite limit as $\rho \to 0$. Due to (2.18) the same limit is taken by $I(\rho, y)$ alone as $\rho \to 0$. We denote this limit by $I(y)$. Its detailed evaluation in Appendix B yields

$$I(y) = -\frac{\pi}{\mu}[(1-\nu)(k_1m_1 + k_2m_2) + k_3m_3]. \tag{2.19}$$

It is useful to introduce new coefficients m_k^* by setting

$$(1-\nu)m_1 = -\mu\sqrt{2}m_1^*, \qquad (1-\nu)m_2 = -\mu\sqrt{2}m_2^*, \qquad m_3 = -\mu\sqrt{2}m_3^* \tag{2.20}$$

this permits us to rewrite (2.19) in the form

$$I(y) = \pi\sqrt{2}(k_1m_1^* + k_2m_2^* + k_3m_3^*). \tag{2.21}$$

The coefficients m_k^* have a simple geometric interpretation. It follows from (2.14) and (2.15) that on C^+

$$\sqrt{\rho}u^* \to m_2^*, \qquad \sqrt{\rho}v^* \to -m_3^*, \qquad \sqrt{\rho}w^* \to m_1^* \quad \text{as} \quad \rho \to 0. \tag{2.22}$$

Henceforth the m_k^* will be referred to as the geometric intensity coefficients of the ordinary fundamental field.

In its restriction to plane strain ($k_3 = 0$, $m_3 = 0$) the result (2.19) was first represented in [2, 3]. Appendix B gives a complete derivation of (2.19) for the convenience of the reader.

Turning now to the configuration of the penny-shaped crack (Fig. 3) we shall take advantage of several coordinate systems. A Cartesian rectangular system with axes x, y and z will be useful when dealing with general aspects of harmonic potentials, describing fields of elastic deformation. The displacements will be denoted by u, v and w and the stresses by σ_x, τ_{xy}, etc. Cylindrical coordinates are convenient for geometric reasons. The

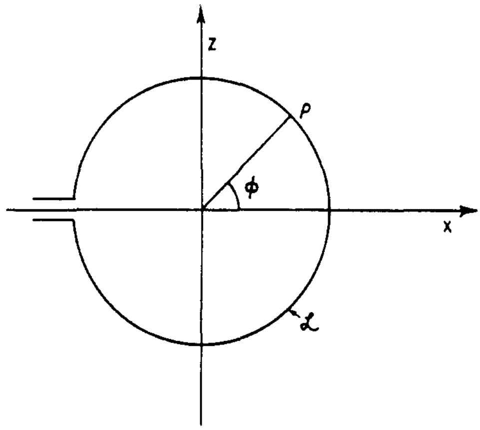


Fig. 5.

relations $x = r\cos\theta$, $y = r\sin\theta$ were already mentioned. We shall have to consider elastic displacements u' in a radial and v' in a tangential direction. These are related to u and v by

$$u' = u\cos\theta + v\sin\theta, \qquad v' = -u\sin\theta + v\cos\theta. \tag{2.23}$$

The stresses associated with cylindrical coordinates will be denoted by σ_r, σ_θ, σ_z; $\tau_{r\theta}$, τ_{rz}, $\tau_{\theta z}$. In order to describe the asymptotic behavior of the field quantities near the edge of the crack we employ local polar coordinates ρ, φ in analogy to their use for the half-plane crack (Fig. 4). But this time we define

$$\rho e^{i\varphi} = r - a + iz. \tag{2.24}$$

In the system of cylindrical coordinates both the first order derivatives of u', v' and w and the ratios u'/r, $-v'/2r$ contribute to the strains. For the asymptotics $\rho \to 0$ the terms u'/r and $v'/2r$ can be neglected so that plane strain deformation with regard to each meridional half-plane θ = constant prevails. (In this context we count a mode III deformation as one of plane strain.) Under these circumstances the asymptotic relations (2.6) and (2.15) of the half-plane crack can be retained after some nominal changes in order to define regular and fundamental fields for the penny-shaped crack. In the displacement formulas one replaces u by the radial and v by the tangential displacement. Stress symbols change by replacing x by r and y by θ in the subscripts; thus σ_x becomes σ_r, etc. All functions of ρ, φ such as α, β, γ_C, γ_S, U_1, U_2 remain the same. Their x-derivatives are formally replaced by r-derivatives. This applies in particular to (2.6)–(2.16). The coefficients k_j, m_k, m_k^* become functions of θ. So do u_0, v_0, w_0. After these modifications the definitions of regular and of ordinary fundamental fields carry over to the structure of the penny-shaped crack. Condition (c) of the ordinary fundamental field should be replaced by (c'): the stresses go to zero as the distance from the center of the crack goes to infinity.

The same rules apply to the integral I in (2.17); u and v change into u' and v' and X and Y into radial and tangential components of traction. Formulas (2.19)–(2.22) retain their meaning accordingly.

It must be emphasized that the result (2.19) is based on the asymptotics for $\rho \to 0$ of both regular and fundamental fields. For this reason, a check on (2.15) must and will be made whenever a candidate for an ordinary fundamental field is under consideration. It will turn out that some special limits of ordinary fundamental fields are regular along certain open arcs of the crack edge. In this case, a check on (2.2) will be added. For this particular check it will suffice to show that relations of the form

$$u^* - u_0 \sim \sqrt{\rho} f_1(\varphi, \theta), \qquad v^* - v_0 \sim \sqrt{\rho} f_2(\varphi, \theta), \qquad w^* - w_0 \sim \sqrt{\rho} f_3(\varphi, \theta) \tag{2.25}$$

exist; (2.25) in turn yields (2.2) and (2.3) by way of the equations of linear elasticity for a field without body forces.

Although the asymptotic relations (2.2) and (2.3) are commonly accepted, the domain of their validity remains to be staked out. What restrictions must be set for the distribution of body forces and boundary tractions in order to ascertain (2.2) and (2.3)? In [3], p. 245, the lesser problem of plane strain deformation without body forces has been dealt with. The difficulties to establish analogous results for states of three-dimensional deformation seem to be formidable. For simplicity we have postulated (2.2) and (2.3) through the concept of the regular field. In similar vein we have introduced ordinary fundamental fields. The construction of such fields for penny-shaped and half-plane cracks will be an important part of this analysis.

3. GENERAL REMARKS ON HARMONIC POTENTIALS OF FUNDAMENTAL FIELDS

Although the penny-shaped crack will be in the center of our analysis, some basic aspects of a more general crack configuration must be mentioned. Let C be a crack of generic shape in the plane $z = 0$. C^+, C^- will denote its faces on upper and lower z-halfspace respectively. We denote by Γ the edge of C and assume that Γ is a smooth contour. The domain of the crack is either the inside or the outside of Γ. We define as an elastic region the three-dimensional cracked space without Γ but with C^+, C^- included and counted as different entities.

In order to describe certain fields of mode I deformation of the elastic region, a Boussinesq–Papkovich potential $G(x, y, z)$ can be employed. G is harmonic, i.e.

$$\nabla^2 G = G_{xx} + G_{yy} + G_{zz} = 0. \tag{3.1}$$

It generates the displacements

$$u = -zG_{xz} - (1-2\nu)G_x, \qquad v = -zG_{yz} - (1-2\nu)G_y, \qquad w = -zG_{zz} + 2(1-\nu)G_z \tag{3.2}$$

and the stresses

$$\begin{aligned} \sigma_x &= -2\mu[(zG_{xx})_z + 2\nu G_{yy}], & \tau_{yz} &= -2\mu z G_{yzz} \\ \sigma_y &= -2\mu[(zG_{yy})_z + 2\nu G_{xx}], & \tau_{xz} &= -2\mu z G_{xzz} \\ \sigma_z &= -2\mu[zG_{zzz} - G_{zz}], & \tau_{xy} &= -2\mu[zG_{xyz} + (1-2\nu)G_{xy}]. \end{aligned} \tag{3.3}$$

The field is without body forces. Assuming continuity in the elastic region of G and its derivatives up to the 3rd order, we may write

$$w = 2(1-\nu)G_z \qquad \text{on} \qquad C^+, C^- \tag{3.4}$$

$$\sigma_z = 2\mu G_{zz}, \qquad \tau_{xz} = 0, \qquad \tau_{yz} = 0 \qquad \text{on} \qquad C^+, C^-. \tag{3.5}$$

In general, the boundary values of G and its derivatives will differ on opposite points of the crack. For mode I, the potential has to be even in z. In this case σ_z has the same value on opposite points of C^+, C^- while w is the same in magnitude but opposite in sign. The potential G has been used in order to determine the field generated by pressure under the crack faces.

It is possible to create mode I deformations in the elastic region by the application of shearing tractions to the crack faces. Due to (3.5), G cannot generate such tractions. G then is not sufficient to describe the class of "regular" fields of mode I in the elastic region.

No such shortcoming of G appears as we endeavour to determine "fundamental" fields of mode I: Ordinary fundamental fields in the elastic region can be defined in analogy to what was done for half-plane and penny-shaped cracks. Briefly speaking, such fields shall have no body forces and no tractions on C^+, C^-; their displacements are permitted to be unbounded in the order $0(d^{-1/2})$ as the edge Γ is approached, d being the distance from Γ. Their stresses shall go to zero as we let $|z|$ go to infinity. The condition of traction-free crack faces means simply

$$G_{zz} = 0 \qquad \text{on} \qquad C^+, C^-, \tag{3.6}$$

and the potential G seems naturally suited for the search of fundamental fields of mode I. Pursuing this goal we should look for functions G with these properties: (a) G is harmonic in the elastic region; G and its derivatives up to the third order are continuous in that region; G is even in z; (b) G satisfies (3.6); (c) $|\text{grad } G| = 0(d^{-1/2})$ as Γ is approached; (d) the stresses (3.3) shall go to zero as we go away from the crack towards infinity.

Condition (3.6) can be rewritten. Combining (3.1) and (3.6) we find

$$\Delta G = G_{xx} + G_{yy} = 0 \qquad \text{on} \qquad C^+, C^-. \tag{3.6*}$$

We can say that G, as a function of x, y on the crack faces must be harmonic in these two variables. In the sequel we shall call a harmonic function G *crack-harmonic* if it also abides by (3.6*). Harmonic functions in two variables have conjugates. This suggests to denote another crack-harmonic function $H = H(x, y, z)$ a conjugate of G if for a suitable choice of $\varepsilon = \pm 1$

$$G_x = \varepsilon H_y, \qquad G_y = -\varepsilon H_x \qquad \text{on} \qquad C. \tag{3.7}$$

If $\varepsilon = 1$ we combine the two potentials into a complex one $J = G + iH$. J satisfies (3.1), i.e. it is a complex-valued harmonic function of x, y, z. In addition the functions $J(x, y, +0)$ and $J(x, y, -0)$ are analytic functions of $x + iy$ in the crack domain so that

$$J_x + iJ_y = 0 \qquad \text{on} \qquad C^+, C^-. \tag{3.8}$$

We call J *crack-analytic*. Such potentials will enhance the search of fundamental fields.

Turning to the other modes of deformation we consider fields generated by three harmonic potentials $g(x, y, z)$, $h(x, y, z)$ and $\psi(x, y, z)$ where

$$\psi_z = g_x + h_y. \tag{3.9}$$

The field quantities are given by

$$u = -2(1-\nu)g + z\psi_x, \qquad v = -2(1-\nu)h + z\psi_y, \qquad w = -(1-2\nu)\psi + z\psi_z. \tag{3.10}$$

$$\begin{aligned} \sigma_x/2\mu &= -2(1-\nu)g_x - 2\nu\psi_z + z\psi_{xx}, & \sigma_z/2\mu &= z\psi_{zz} \\ \sigma_y/2\mu &= -2(1-\nu)h_y - 2\nu\psi_z + z\psi_{yy}, & \tau_{xy}/2\mu &= -(1-\nu)(g_y + h_x) + z\psi_{xy} \\ \tau_{xz}/2\mu &= -(1-\nu)g_z + \nu\psi_x + z\psi_{xz}, & \tau_{yz}/2\mu &= -(1-\nu)h_z + \nu\psi_y + z\psi_{yz}. \end{aligned} \tag{3.11}$$

They form a field without body forces. The preceding formulas are well known, especially for their use in crack analysis. The reader should observe that the functions g and h of (3.9)–(3.11) appear in the form of z-derivatives g_z, h_z in [6, 16]. Assuming the continuity of the potentials and of their derivatives up to second order in the elastic region, we find in particular

$$u = -2(1-\nu)g, \qquad v = -2(1-\nu)h, \qquad w = -(1-2\nu)\psi \qquad \text{on} \qquad C^+, C^-; \tag{3.12}$$

$$\sigma_z = 0, \qquad \tau_{xz}/2\mu = -(1-\nu)g_z + \nu\psi_x, \qquad \tau_{yz}/2\mu = -(1-\nu)h_z + \nu\psi_y \qquad \text{on} \qquad C^+, C^-. \tag{3.13}$$

For the shearing modes II and III one assumes g and h to be odd in z; ψ has to be even. The potentials g, h and ψ can be used to analyse fields generated by shearing tractions on the crack faces, such that opposite points have opposite tractions, but fields of modes II and III can also be generated by normal tractions. As (3.13) shows, such fields are beyond a description by g, h, ψ. So much for "regular" fields.

In the case of ordinary fundamental fields, already defined above, there must be no traction on the crack. This leads to the two conditions:

$$-(1-\nu)g_z+\nu\psi_x = 0, \qquad -(1-\nu)h_z+\nu\psi_y = 0 \quad \text{on} \quad C^+, C^- \tag{3.14}$$

and suggests that the potentials g, h and ψ are ideally suited for the search of fundamental fields.

Let G be crack-analytic. We define

$$g = G_z, \qquad h = iG_z, \qquad \psi = G_x+iG_y. \tag{3.15}$$

Condition (3.9) is satisfied. On the crack ψ vanishes. Furthermore

$$ih_z = -g_z = -G_{zz} = G_{xx}+G_{yy} = 0 \qquad \text{on} \qquad C^+, C^-. \tag{3.16}$$

Consequently the real parts of g, h and ψ describe a field with no body forces and with no tractions on the crack faces; the same holds for the imaginary parts.

Let L be another complex-valued harmonic function such that L_z is crack-analytic. Now we define

$$g = (1-\nu)L_x+iL_y, \qquad h = (1-\nu)L_y-iL_x, \qquad \psi = -(1-\nu)L_z. \tag{3.17}$$

We find

$$g_x+h_y-\psi_z = (1-\nu)\nabla^2 L = 0. \tag{3.18}$$

Again (3.9) is satisfied. On the crack we have $L_{zx}+iL_{zy} = 0$ and

$$\begin{aligned} -(1-\nu)g_z+\nu\psi_x &= -(1-\nu)[(1-\nu)L_{xz}+iL_{yz}+\nu L_{zx}] \\ &= -(1-\nu)(L_{zx}+iL_{zy}) = 0 \\ -(1-\nu)h_z+\nu\psi_y &= -(1-\nu)[(1-\nu)L_{yz}-iL_{xz}+\nu L_{yz}] \\ &= i(1-\nu)(L_{zx}+iL_{zy}) = 0. \end{aligned} \tag{3.19}$$

Conditions (3.14) are satisfied. Altogether the real parts of the potentials (3.17) describe a field without body forces and without tractions on C^+, C^-; so do the imaginary parts.

The results associated with (3.17) and (3.15) reduce the search for ordinary fundamental fields of modes II and III to the acquisition of crack-analytic potentials. In the case of (3.15), G should be even in z, while the L of (3.17) should be odd. In both cases $|\text{grad } G|$ and $|\text{grad } L|$ shall be of the order $0(d^{-1/2})$ as Γ is approached. We can expect that any G that yields an ordinary fundamental field of mode I gives rise to a field of mixed modes II and III through (3.15). As for L, one can conjecture that it should be possible to relate it to a suitable G of mode I. This is indeed so for the case of the penny-shaped and the half-plane crack.

Let G be even in z and crack-analytic. For the half-plane crack of Fig. 4 we introduce

$$L = 2(zG_x-xG_z). \tag{3.20}$$

It is easily checked and explicitly shown in Appendix A that L is harmonic. It is also odd in z. Since $x = 0$, $z = 0$ hold on the edge of the crack one can expect $|\text{grad } L|$ to have the same order of growth as $|\text{grad } G|$ as the edge is approached. We observe that

$$L_z = 2G_x - 2xG_{zz} = 2G_x \qquad \text{on} \qquad C^+, C^-. \tag{3.21}$$

Now if $G(x, y, +0)$ and $G(x, y, -0)$ are analytic in $x+iy$ so are their x-derivatives. L_z then is crack-analytic.

For the penny-shaped crack we define L by

$$aL = (z^2 + a^2 - r^2)G_z + 2rzG_r + zG. \tag{3.22}$$

As shown in Appendix A, the function L is harmonic; it is odd in z and $|\text{grad } L|$ can be expected to grow like $|\text{grad } G|$ as the edge $r = a$ is approached. On the crack we have

$$aL_z = G + 2rG_r = G + 2(xG_x + yG_y) = G + 2(x+iy)G_x \tag{3.23}$$

where the right-hand side is analytic in $(x+iy)$. The function L of (3.22) is thus seen to have a crack-analytic derivative L_z.

For easy reference the field defined by (3.15) will be denoted as a field of the first kind; the field by (3.17) will be referred to as a field of the second kind. We use the same names in order to distinguish the displacements of these fields.

The foregoing considerations indicate that the crack-analytic potentials G of mode I are the key to all of our goals. With their aid we shall be able to find all relevant ordinary fundamental fields for mode I for penny-shaped and half-plane crack. With their aid we shall construct ordinary fundamental fields of the first kind for the shearing modes; finally, by way of (3.22) or (3.20), a potential G of mode I lets us construct fields of the second kind for the shearing modes.

We conclude this section with a list of representations of u', v' and w in cylindrical coordinates. It should be kept in mind that all potentials are complex-valued and that the displacements derived from them are actually pairs of displacement in complex form.

Mode I

$$\begin{aligned} u' &= -zG_{rz} - (1-2\nu)G_r = (G - zG_z)_r - 2(1-\nu)G_r \\ w &= -zG_{zz} + 2(1-\nu)G_z = (G - zG_z)_z + 2(1-\nu)G_z \\ v' &= -zG_{\theta z}/r - (1-2\nu)G_\theta/r. \end{aligned} \tag{3.24}$$

Modes II and III (first kind)

$$\begin{aligned} u' &= -2(1-\nu)e^{i\theta}G_z + z\psi_r \qquad \text{with} \qquad \psi = e^{i\theta}(G_r + iG_\theta/r) \\ v' &= -2i(1-\nu)e^{i\theta}G_z + z\psi_\theta/r, \qquad w = -2(1-\nu)\psi + (z\psi)_z. \end{aligned} \tag{3.25}$$

Modes II and III (second kind)

$$\begin{aligned} u' &= -2(1-\nu)[(1-\nu)L_r + iL_\theta/r] + z\psi_r \qquad \text{with} \qquad \psi = -(1-\nu)L_z \\ v' &= -2(1-\nu)[-iL_r + (1-\nu)L_\theta/r] + z\psi_\theta/r \\ w &= -2(1-\nu)\psi + (z\psi)_z. \end{aligned} \tag{3.26}$$

4. THE PENNY-SHAPED CRACK. FUNDAMENTAL FIELDS OF MODE I

A class of ordinary fundamental fields has already been described in [4]. We repeat and extend the former results. In this context it is necessary to introduce oblate spheroidal coordinates s, t, given by

$$\begin{gathered} r+iz = a\cosh(s+it) \qquad \text{in complex,} \\ r = a\cosh s\cdot\cos t, \qquad z = a\sinh s\cdot\sin t \qquad \text{in real form} \end{gathered} \tag{4.1}$$

s, t have ranges $0 \leqslant s < \infty$; $-\frac{1}{2}\pi \leqslant t \leqslant \frac{1}{2}\pi$. On the crack $s = 0$. t is positive for $z > 0$ and negative for $z < 0$; $t = 0$ for $z = 0$ and $r > a$. Our first concern are harmonic functions of the form

$$G(x,y,z) = H(q,s) \qquad \text{with} \qquad q = \frac{r}{a}\exp i(\theta-\theta'), \tag{4.2}$$

where θ' is some parametric angle. Now

$$\nabla^2 G = H_q\cdot\nabla^2 q + H_s\nabla^2 s + (q_x^2+q_y^2)H_{qq} + 2(s_x q_x + s_y q_y)H_{qs} + (s_x^2+s_y^2+s_z^2)H_{ss}. \tag{4.3}$$

Here the co-factors of H_q, H_{qq} vanish (trivial); as for the other terms we list for present and later use the relations

$$\begin{gathered} s_r = t_z = \sinh s\cdot\cos t/aN, \qquad s_z = -t_r = \cosh s\cdot\sin t/aN \\ \text{with} \qquad N = \sinh^2 s + \sin^2 t \end{gathered} \tag{4.4}$$

$$\begin{gathered} s_x^2+s_y^2+s_z^2 = s_r^2+s_z^2 = 1/a^2N \\ \nabla^2 s = s_r/r = (\tanh s)/a^2N;\ s_x q_x + s_y q_y = q s_r/r. \end{gathered} \tag{4.5}$$

Altogether

$$a^2N\cdot\nabla^2 G = \tanh s(H_s + 2qH_{qs}) + H_{ss}. \tag{4.6}$$

Harmonic functions $H(q, s)$ satisfy

$$H_{ss} + \tanh s\cdot(H_s + 2qH_{qs}) = 0. \tag{4.7}$$

For nonnegative integers n we set up

$$G_n(x,y,z) = q^n H_n(s). \tag{4.8}$$

Harmonicity is achieved if

$$H_n''(s) + (2n+1)\tanh s\cdot H_n'(s) = 0 \tag{4.9}$$

whence

$$H_n'(s) = -\frac{c_0}{\cosh^{2n+1} s}. \tag{4.10}$$

We give c_0 the same value for all n, namely

$$c_0 = -\frac{\ell^{5/2}}{\sqrt{2a}(1-\nu)} \tag{4.10a}$$

where ℓ is the unit of length; this gives c_0 the dimension of the square of length. We define

$$H_n(s) = c_0\int_s^\infty \frac{d\sigma}{\cosh^{2n+1}\sigma}. \tag{4.10b}$$

The integration yields

$$H_0(s) = c_0[\pi/2 - \arctan(\sinh s)] \tag{4.11}$$

and, more generally,

$$H_n(s) = c_n\left[\pi/2 - \arctan(\sinh s) - c_0 \sinh s \sum_{k=1}^{n} \frac{1}{2kc_k \cosh^{2k} s}\right] \tag{4.12}$$

with

$$c_k = c_0(-1)^k \binom{-(1/2)}{k} \qquad \text{for} \qquad k \geqslant 1.$$

Since (4.12) will play no major role in the sequel the verification of that representation is left to the interested reader.

We have to look into the behavior of G_n near the crack edge and far away from it. In this context, we list here

$$\begin{aligned} \rho e^{i\varphi} &= a[\cosh(s+it) - 1] = 2a \sinh^2 \tfrac{1}{2}(s+it) \\ 2\sqrt{a} \sinh \tfrac{1}{2}(s+it) &= (2\rho e^{i\varphi})^{1/2} = \alpha + i\beta \end{aligned} \tag{4.13}$$

whence

$$\sqrt{a}(s+it) = \alpha + i\beta - \frac{(\alpha+i\beta)^3}{24a} + 0(\rho^{5/2}) \qquad \text{as} \qquad \rho \to 0. \tag{4.14}$$

We observe that

$$a^2 \sinh^2 s < r^2 + z^2 = R^2 < a^2 \cosh^2 s \tag{4.15}$$

which entails

$$2R = ae^s + 0(e^{-s}) \qquad \text{as} \qquad s \to \infty \tag{4.16}$$

$$ae^s = 2R + 0(1/R) \qquad \text{as} \qquad R \to \infty. \tag{4.16a}$$

The power q^n is analytic in $(x+iy)$; since $G_n(x, y, 0) = q^n H_n(0)$ on the crack the potential G_n is crack-analytic. Due to (4.10b) the functions $H_n(s)$ are negative. Another consequence of (4.10b) is

$$0 < -H_0(s) < -2c_0 \cdot e^{-s} \tag{4.17}$$

and

$$0 < -H_n(s) < -\frac{H_0(s)}{\cosh^{2n} s}. \tag{4.18}$$

From (4.18) it follows that

$$|G_n| < -\left(\frac{|q|}{\cosh^2 s}\right)^n \cdot G_0 = -\left(\frac{\cos t}{\cosh s}\right)^n G_0. \tag{4.19}$$

From (4.16)–(4.19) we infer that

$$G_n = 0(R^{-n-1}) \qquad \text{as} \qquad R \to \infty. \tag{4.20}$$

As a Boussinesq–Papkovich potential G_n generates the displacements (3.2) and the stresses (3.3). For $R \to \infty$ the displacements have the order $0(R^{-n-2})$ while the stresses go to zero like R^{-n-3}. This follows from (4.4), (4.9) and (4.10) and from (4.16)–(4.20).

By (4.14) s and t go to zero as $\rho \to 0$. We may write

$$H_n(s) = H_n(0) - c_0 s + 0(s^3) = H_n(0) + c\alpha + 0(\rho^{3/2}) \qquad \text{with} \qquad c = -c_0/\sqrt{a}. \tag{4.21}$$

Any unboundedness of grad G_n can only be due to the term $c\alpha$. For the simple case $G = c\alpha$ the formulas (3.24) for the displacements yield $v' = 0$ and

$$u'/c = (\alpha - z\alpha_z)_r - 2(1-\nu)\alpha_r, \qquad w/c = (\alpha - z\alpha_z)_z + 2(1-\nu)\alpha_z. \tag{4.22}$$

With reference to (2.12)–(2.13a) we use $\alpha_z = -\beta_r$ and

$$\alpha - z\alpha_z = \alpha - \tfrac{1}{2}\beta \sin\varphi = (1 - \sin^2\varphi/2)\alpha = \tfrac{1}{2}(1+\cos\varphi)\alpha = U_{1r}$$

so that (4.22) can be rewritten as

$$u'/c = [U_{1r} - 2(1-\nu)\alpha]_r, \qquad w/c = [U_{1z} - 2(1-\nu)\beta]_r. \tag{4.22a}$$

The displacements u', v' and w comply with the asymptotics (2.15) of an ordinary fundamental field for the approach $\rho \to 0$. In the preceding form the displacements are those associated with the potential G_0. In the case of G_n the factor q^n must be taken into account. For $n \geqslant 1$ the displacement v' will no longer vanish identically but it will stay bounded. The other two displacements are obtained from (4.22a) by multiplying the right-hand sides by q^n for $r = a$, i.e. by $\exp ni(\theta - \theta')$. We can now pronounce that G_n generates, through (3.2) and (3.3), a fundamental field. It has the geometric intensity coefficients

$$m_1^* = \frac{1}{a}\exp ni(\theta - \theta'),\dagger \qquad m_2^* = 0, \qquad m_3^* = 0. \tag{4.23}$$

Actually we have two fields, corresponding to $Re\, G_n$ and $Im\, G_n$. The intensity coefficients follow suit.

Let a regular field, not necessarily of mode I, be the response to a distribution of body forces **F** and of tractions **T** on C^+, C^-. Its intensity factors k_j are functions $k_j(\theta)$ of the position angle θ on the edge of the crack. Let

$$k_1(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \tag{4.24}$$

represent k_1. The Fourier coefficients are given by

$$2\pi a_0 = \int_0^{2\pi} k_1(\theta)\, d\theta, \qquad \pi a_n = \int_0^{2\pi} k_1(\theta) \cos n\theta\, d\theta$$
$$\pi b_n = \int_0^{2\pi} k_1(\theta) \sin n\theta\, d\theta \tag{4.25}$$

† The factor $\ell^{5/2}$, where ℓ denotes unit length, has been omitted.

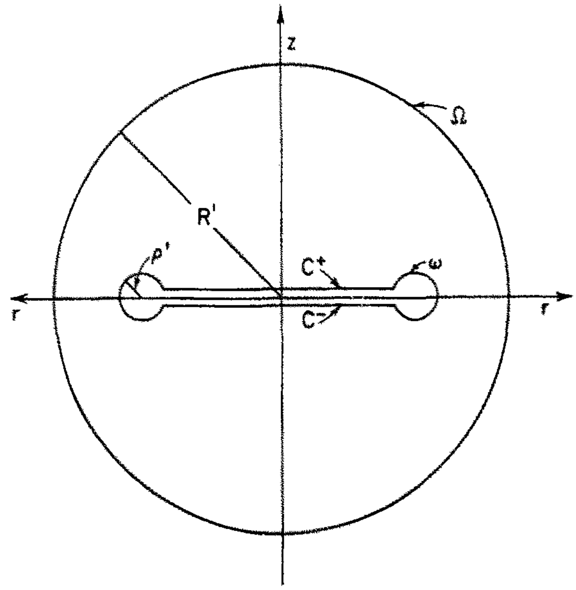


Fig. 6.

so that

$$k_1(\theta') = \frac{1}{2\pi}\int_0^{2\pi} k_1(\theta)\,d\theta + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi} k_1(\theta)\cos n(\theta-\theta')\,d\theta. \tag{4.26}$$

Since $k_1(\theta)$ is real-valued we may rewrite (4.26) in the form

$$k_1(\theta') = Re\left\{\frac{1}{2\pi}\int_0^{2\pi} k_1(\theta)\,d\theta + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi} k_1(\theta)\exp ni(\theta-\theta')\,d\theta\right\}. \tag{4.27}$$

We shall determine the integrals in (4.27) with the aid of the fundamental fields generated by the potentials G_n. To this end we apply the reciprocity theorem to the regular field and to the pair of fields defined by G_n within the subregion V' of the configuration of Fig. 3. This subregion, shown in Fig. 6, satisfies both $R \leqslant R' < \infty$ and $\rho \geqslant \rho' > 0$. The boundary of V' consists of the torus $\rho = \rho'$, of the sphere $R = R'$ and of certain portions of C^+, C^-. For easy reference we denote the torus by ω and the remaining boundary by S'. In V' the regular field yields tractions $\mathbf{T}$ on ω and on S'; we denote its displacement vector by W. G_n generates, through (3.2), a complex-valued displacement vector field W_n^*, and through (3.3) complex-valued tractions on the boundary of V', which we denote by $\mathbf{T}_n^*$. We repeat that G_n does not generate body forces. The reciprocity theorem leads to the following balance of energies:

$$\int_\omega [(W,\mathbf{T}_n^*)-(W_n^*,\mathbf{T})]\,dS = \int_{S'} [(W_n^*,\mathbf{T})-(W,\mathbf{T}_n^*)]\,dS + \int_{V'} (W_n^*,\mathbf{F})\,dV. \tag{4.28}$$

Although the vectors marked by an asterisk are complex-valued the scalar product notation in (4.28) follows the definition for real-valued vectors. Thus:

$$(W,\mathbf{T}'+i\mathbf{T}'') = (\mathbf{T}'+i\mathbf{T}'',W) = (\mathbf{T}',W)+i(\mathbf{T}'',W)$$

for real $\mathbf{T}'$, $\mathbf{T}''$. We let $\rho' \to 0$. Asymptotically $dS = a\rho'\,d\varphi\,d\theta$ on ω; the integration on ω with respect to φ gives

$$I(\rho',\theta) = \rho'\int_{-\pi}^{\pi} [(W,\mathbf{T}_n^*)-(W_n^*,\mathbf{T})]\,d\varphi$$

which has the form (2.17). The asymptotic behavior of regular and ordinary fundamental fields led us to (2.21) which we repeat here as

$$\lim_{\rho'\to 0} I(\rho',\theta) = I(\theta) = \pi\sqrt{2}(k_1 m_1^* + k_2 m_2^* + k_3 m_3^*)$$

In the case in hand, the geometric intensity coefficients are those of (4.23). Altogether

$$\lim_{\rho'\to 0} \int_\omega [\ldots]\,\mathrm{d}S = \pi\sqrt{2}\int_0^{2\pi} k_1(\theta)\exp ni(\theta-\theta')\,\mathrm{d}\theta. \tag{4.29}$$

As $\rho' \to 0$, the subregion V' changes into the intersection of the original elastic region V with the ball $R \leqslant R'$. S' becomes the union of C^+, C^- and the sphere Ω of radius R'. Since G_n has no tractions on the crack faces the right-hand side of (4.28) will balance the limit (4.29) in the form

$$\pi\sqrt{2}\int_0^{2\pi} k_1(\theta)\exp ni(\theta-\theta')\,\mathrm{d}\theta = \int_C (W_n^*, \mathbf{T})\,\mathrm{d}S + \int_\Omega [(W_n^*, \mathbf{T}) - (W, \mathbf{T}_n^*)]\,\mathrm{d}S + \int_{V'} (W_n^*, \mathbf{F})\,\mathrm{d}V.\dagger \tag{4.30}$$

Finally we let $R' \to \infty$. We can expect

$$\pi\sqrt{2}\int_0^{2\pi} k_1(\theta)\exp ni(\theta-\theta')\,\mathrm{d}\theta = \int_C (W_n^*, \mathbf{T})\,\mathrm{d}S + \int_V (W_n^*, \mathbf{F})\,\mathrm{d}V. \tag{4.31}$$

This happens in particular if, for $R \to \infty$, W is bounded, $\mathbf{T} \to 0$, $\mathbf{F} = 0(R^{-\lambda})$ for some $\lambda > 1$. With (4.27) and (4.31) we have determined $k_1(\theta')$ for the given regular field.

It is possible to give the sum in (4.27) in closed form. We compose

$$W^* = W_0^* + 2\sum_{n=1}^{\infty} W_n^* \tag{4.32}$$

as well as

$$G^* = G_0 + 2\sum_{n=1}^{\infty} G_n. \tag{4.33}$$

Due to (4.19) the series (4.33) converges uniformly in domains in which ρ is bounded away from zero. The same applies to its partial derivatives of any order and to (4.32) in particular. G^* is a function of q and s. Taking the s-derivative at fixed q we find

$$G_s^* = -\frac{c_0}{\cosh s}\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{q}{\cosh^2 s}\right)^n\right] = \frac{-c_0}{\cosh s}\left(-1 + \frac{2\cosh^2 s}{\cosh^2 s - q}\right). \tag{4.34}$$

Integration in accord with (4.10b) yields

$$G^* = -c_0\left\{\frac{1}{\sqrt{q-1}}\log\frac{\sinh s - \sqrt{q-1}}{\sinh s + \sqrt{q-1}} + \pi/2 - \arctan(\sinh s)\right\}. \tag{4.35}$$

† Here and elsewhere integration over C applies to C^+ and to C^-.

Here we determine the square root such that

$$Im\sqrt{q-1} \geqslant 0 \tag{4.35a}$$

and the log-function such that

$$\pi > Im \log (\ldots) \geqslant -\pi. \tag{4.35b}$$

G^* is a complex-valued harmonic potential. It takes the values

$$G^* = c_0\left(\frac{\pi i}{\sqrt{q-1}} - \pi/2\right) \quad \text{on} \quad C^+, C^-. \tag{4.36}$$

Evidently G^* is crack-analytic. The two fields generated by G^* are the limits of ordinary fundamental fields. As such they need not be ordinary fundamental. We must take a closer look at the behavior of G^* as $R \to \infty$ and as $\rho \to 0$. For $R \to \infty$, equivalent with $s \to \infty$, we have $\sqrt{q-1}/\sinh s \to 0$. In this case (4.35) yields

$$G^* = \frac{c_0}{\sinh s}[1+0(e^{-2s})] \tag{4.37}$$

which coincides with the asymptotic behavior of G_0, as one would expect.

For the approach $\rho \to 0$ we assume some $\varepsilon > 0$ and

$$|e^{i\theta} - e^{i\theta'}| \geqslant \varepsilon. \tag{4.38}$$

By this condition q is bounded away from unity and $(\sinh s)/\sqrt{q-1}$ goes to zero as ρ does. From (4.35) we derive

$$G^* = f_0(q) + f_1(q) \sinh s + f_3(q) \sinh^3 s + 0(\sinh^5 s) \tag{4.39}$$

with

$$f_0(q) = c_0\left(-\pi/2 + \frac{i\pi}{\sqrt{q-1}}\right), \qquad f_1(q) = c_0\frac{q+1}{q-1}$$
$$3f_3(q) = c_0\frac{2q}{(q-1)^2} - f_1(q) = -\frac{\partial}{\partial r}(rf_1). \tag{4.39a}$$

The behavior of the fields of G^* near the crack edge and under the restriction (4.38) is determined by the approximation

$$G^* = c_0\frac{q+1}{q-1}s. \tag{4.40}$$

This indicates that the fields of G^* behave like ordinary ones everywhere at the edge but at the point $r = a$, $\theta = \theta'$. We shall go into more detail in Section 6.

Turning to W^*, the displacement vector generated by G^*, we derive from (4.27), (4.31), (4.32) and (4.33)

$$k_1(\theta') = \frac{1}{2\sqrt{2}\pi^2}\left[\int_C (Re\ W^*, \mathbf{T})\ \mathrm{d}S + \int_V (Re\ W^*, \mathbf{F})\ \mathrm{d}V\right]. \tag{4.41}$$

With this formula our goal has been reached with respect to mode I. The combination

(4.35) and (4.41) represents a concrete example for the representation of a stress intensity factor due to generic three-dimensional loading **F**, **T**. The search for such representations was suggested by Rice[11, Appendix].

5. THE SHEARING MODES

We begin with the displacements of the first kind as listed under (3.25). For $G = G_n = q^n H_n(s)$ we find

$$G_z = q^n H_n'(s) s_z, \qquad \psi = e^{i\theta} \cdot q^n H_n'(s) s_r. \tag{5.1}$$

For the approach to the edge of the crack the terms s_z, s_r will determine the asymptotic behavior of the displacements. In this context we can replace $H_n'(s)$ by $-c_0$ and q^n by $\exp ni(\theta-\theta')$. This leads us to

$$u' \sim c_0 E_1[2(1-\nu)s_z - zs_{rr}], \quad v' \sim 2ic_0 E_1(1-\nu)s_z, \quad w \sim c_0 E_1[2(1-\nu)s_r - (zs_r)_z] \\ \text{where} \quad E_1 = \exp\,[i\theta' + (n+1)i(\theta-\theta')]. \tag{5.2}$$

By (4.14)

$$\sqrt{a}s \sim \alpha, \qquad \sqrt{a}s_r \sim \alpha_r, \qquad \sqrt{a}s_z \sim -\beta_r \tag{5.3}$$

so that

$$\sqrt{a}u' \sim c_0 E_1 \cdot [-2(1-\nu)\beta - (z\alpha)_r]_r, \qquad \sqrt{a}v' \sim ic_0 E_1(1-\nu)\beta/\rho \\ \sqrt{a}w \sim c_0 E_1[2(1-\nu)\alpha - (z\alpha)_z]_r. \tag{5.4}$$

A comparison with (2.15) shows that these displacements represent shearing modes with the geometric intensity coefficients

$$m_1^* = 0, \qquad m_2^* = -E_1/a, \qquad m_3^* = iE_1/a. \tag{5.5}$$

The asymptotics for $R \to \infty$ of the field of the first kind are easy to establish. The properties of G_n imply that the displacements go to zero like R^{-n-2}, while the stresses are of the order $0(R^{-n-3})$. Altogether we can now pronounce that the fields of the first kind due to G_n are ordinary fundamental.

Turning to the field of the second kind we derive from G the potential L through (3.22). If $G = H(q, s)$ then

$$L = \sin t[H_s \cosh s + (H + 2qH_q) \sinh s]. \tag{5.6}$$

This relation is easily proved with the aid of (4.5) and

$$a^2 + z^2 - r^2 + 2rz \cdot s_r/s_z = a^2 N. \tag{5.7}$$

The details are left to the reader. The special case $G = G_n$ yields

$$L_n = q^n[\cosh sH_n'(s) + (2n+1) \sinh sH_n(s)] \cdot \sin t. \tag{5.8}$$

For the asymptotics of $\rho \to 0$, one can use

$$L_n \sim -c_0 E_2 t, \qquad E_2 = \exp ni(\theta - \theta'). \tag{5.8a}$$

Now (4.14) states $\sqrt{a} \cdot t \sim \beta$; this and the relations (2.12) and (2.12a) yield the following asymptotic representations of the displacements of second kind:

$$\begin{aligned}\sqrt{a}\cdot u' &\sim (1-\nu)c_0E_2[2(1-\nu)\beta_r+(z\beta_z)_r] = (1-\nu)c_0E_2[2(1-\nu)\beta+(z\alpha)_r]_r\\ \sqrt{a}\cdot v' &\sim -2i(1-\nu)c_0E_2\beta_r = i(1-\nu)c_0E_2\beta/\rho \\ \sqrt{a}\cdot w &\sim (1-\nu)c_0E_2[-2(1-\nu)\alpha+(z\alpha)_z]_r.\end{aligned} \tag{5.9}$$

This again characterizes an ordinary fundamental field of mixed modes II and III, this time with the geometric intensity coefficients

$$m_1^* = 0, \qquad m_2^* = (1-\nu)E_2/a, \qquad m_3^* = iE_2/a. \tag{5.10}$$

Let $R \to \infty$. From (5.8) and the properties of H_n it follows that

$$L_n = 0(R^{-n-2}). \tag{5.11}$$

Displacements and stresses of the field of the second kind due to L_n go to zero as $R \to \infty$. They decay one order faster than the corresponding quantities of the field of the first kind generated by G_n. We can now state that the fields of the second kind for $L = L_0, L_1, \ldots$ are ordinary fundamental.

It is convenient to denote the displacement vectors of the field of the first kind by $\mathscr{D}_1G$, if G is the generating potential. We denote the displacement vectors of the second kind by $\mathscr{D}_2G$, if L is derived from G by means of (3.22). If $G = G_n$ then the θ-dependence of the displacements shows up in a factor $E(\theta)$, common to all displacements of the same field. Obviously $E(\theta) = \text{constant} \exp ni\theta$ for $\mathscr{D}_2G_n$, and $E(\theta) = \text{constant} \exp (n+1)i\theta$ for $\mathscr{D}_1G_n$. Without loss of generality we may set $E = E_1$ for the first and $E = E_2$ for the second kind. E_1 and E_2 appear in (5.2) and (5.8a) respectively.

The field $\mathscr{D}_2G_0$ deserves special attention. $G_0 = H_0(s)$ is real-valued, and so is L_0. The real part of $\mathscr{D}_2G_0$ is represented by

$$\begin{aligned} u' &= -(1-\nu)\cdot[2(1-\nu)L_{0r}+z\cdot L_{0rz}], \qquad v' = 0 \\ w' &= -(1-\nu)\cdot[-2(1-\nu)L_{0z}+(zL_{0z})_z]\end{aligned} \tag{5.12}$$

and the imaginary part by

$$v' = 2(1-\nu)L_{0r}, \qquad u' = 0, \qquad w = 0. \tag{5.13}$$

The results (5.5) and (5.10) permit to combine the fields of first and second kind so as to obtain fields of pure modes II and III. One finds the following combinations:

$$W_0^* = (Re\,\mathscr{D}_2G_0)/(1-\nu); \qquad W_n^* = (\mathscr{D}_2G_n - e^{-i\theta'}\cdot\mathscr{D}_1G_{n-1})/(2-\nu) \qquad n = 1, 2, \ldots \tag{5.14}$$

for mode II;

$$W_0^* = Im\,\mathscr{D}_2G_0; \qquad W_n^* = -i(\mathscr{D}_2G_n + (1-\nu)e^{-i\theta'}\cdot\mathscr{D}_1G_{n-1})/(2-\nu) \qquad n = 1, 2, \ldots \tag{5.15}$$

for mode III.

In the case (5.14) the geometric intensity coefficients m_1^* and m_3^* vanish; we have $am_2^* = \exp ni(\theta-\theta')$ for W_n^* and all n. In the case (5.15) m_2^*, m_3^* exchange roles.

Relations (4.27), (4.29) and (4.31) apply again after certain nominal changes. They yield $k_2(\theta')$ instead of k_1 if the displacements come from (5.14) and if the $\mathbf{T}_n^*$ follows suit. Similarly they furnish $k_3(\theta')$ if the W_n^* are from (5.15). Ultimately we want the analogues of (4.41) for k_2 and k_3; this is easy to do. We extend the meaning of the composition (4.32).

If the W_n^* in (4.32) are of mode I (as originally assumed) we set $W^* = W_{\rm I}^*$ for the sum. If the W_n^* come from (5.14) we set $W^* = W_{\rm II}^*$. If they come from (5.15) the sum W^* is denoted by $W_{\rm III}^*$. In detail we have

$$W_{\rm II}^* = (Re\,\mathscr{D}_2 G_0)/(1-\nu) + 2\sum_{n=1}^{\infty} (\mathscr{D}_2 G_n - e^{-i\theta'} \cdot \mathscr{D}_1 G_{n-1})/(2-\nu)$$
$$W_{\rm III}^* = Im\,\mathscr{D}_2 G_0 - 2i \sum_{n=1}^{\infty} [\mathscr{D}_2 G_n + (1-\nu)e^{-i\theta'} \cdot \mathscr{D}_1 G_{n-1}]/(2-\nu). \tag{5.16}$$

Through (3.22), G^* gives rise to a potential L^*. From (4.35) and (5.6) we derive

$$L^* = \left[c_0 \frac{q+1}{q-1} - \frac{q \sinh s}{q-1} G_0 - \frac{\sinh s}{q-1} G^* \right] \cdot \sin t. \tag{5.17}$$

L^* determines a field of the second kind with displacements $\mathscr{D}_2 G^*$. With the aid of G^*, the sums (5.16) can be expressed as follows:

$$(2-\nu) W_{\rm II}^* = \frac{2-\nu}{1-\nu} Re\,\mathscr{D}_2 G_0 + \mathscr{D}_2 (G^* - G_0) - e^{-i\theta'} \cdot \mathscr{D}_1 (G^* + G_0) \tag{5.18}$$

$$(2-\nu) W_{\rm III}^* = (2-\nu) Im\,\mathscr{D}_2 G_0 - i\mathscr{D}_2 (G^* - G_0) - i(1-\nu) e^{-i\theta'} \cdot \mathscr{D}_1 (G^* + G_0). \tag{5.19}$$

These expressions yield in particular

$$(2-\nu) Re\, W_{\rm II}^* = Re\,\{\mathscr{D}_2 G_0/(1-\nu) + \mathscr{D}_2 G^* - e^{-i\theta'} \cdot \mathscr{D}_1 (G^* + G_0)\} \tag{5.18a}$$

$$(2-\nu) Re\, W_{\rm III}^* = Im\{(1-\nu)\mathscr{D}_2 G_0 + \mathscr{D}_2 G^* + (1-\nu) e^{-i\theta'} \mathscr{D}_1 (G^* + G_0)\}. \tag{5.19a}$$

The required analogues of (4.41) are:

$$k_2(\theta') = \frac{1}{2\sqrt{2\pi^2}} \left[\int_C (Re\, W_{\rm II}^*, \mathbf{T})\, \mathrm{d}S + \int_V (Re\, W_{\rm II}^*, \mathbf{F})\, \mathrm{d}V \right] \tag{5.20}$$

$$k_3(\theta') = \frac{1}{2\sqrt{2\pi^2}} \left[\int_C (Re\, W_{\rm III}^*, \mathbf{T})\, \mathrm{d}S + \int_V (Re\, W_{\rm III}^*, \mathbf{F})\, \mathrm{d}V \right]. \tag{5.21}$$

Here the weight functions $Re\, W_{\rm II}^*$, $Re\, W_{\rm III}^*$ are available in closed form through (5.18a) and (5.19a).

The fundamental fields with the displacements $W_{\rm II}^*$ and $W_{\rm III}^*$ are important beyond the configuration of the penny-shaped crack. For this reason we list their potentials g, h and ψ as well. We distinguish them by an asterisk.

Mode II

$$(2-\nu) g^* = -e^{-i\theta'} \cdot (G^* + G_0)_z + L_{0r} e^{-i\theta} + (1-\nu) L_x^* + i L_y^*$$
$$(2-\nu) h^* = -i e^{-i\theta'} \cdot (G^* + G_0)_z + i L_{0r} e^{-i\theta} + (1-\nu) L_y^* - i L_x^*$$
$$(2-\nu) \psi^* = -e^{i(\theta-\theta')} G_{0r} - e^{-i\theta'} (G_x^* + i G_y^*) - L_{0z} - (1-\nu) L_z^*.$$

Mode III

$$(2-\nu) g^* = -i(1-\nu) e^{-i\theta'} (G^* + G_0)_z + i(1-\nu) L_{0r} e^{-i\theta} - i(1-\nu) L_x^* + L_y^*$$
$$(2-\nu) h^* = (1-\nu) e^{-i\theta'} (G^* + G_0)_z - (1-\nu) L_{0r} e^{-i\theta} - L_x^* - i(1-\nu) L_y^*$$
$$(2-\nu) \psi^* = -i(1-\nu)[e^{i(\theta-\theta')} G_{0r} + e^{-i\theta'} (G_x^* + i G_y^*) + (L_0 - L^*)_z].$$

G^* and L^* have been referred to Cartesian coordinates. Since G_0 and L_0 depend on r and z only, their derivatives have been given accordingly. One can rewrite the lists of g^*, h^* and ψ^* in terms of cylindrical coordinates. In this case it is useful to present the information about these potentials in the following form:

Mode II

$$g^{**} = g^* \cos\theta + h^* \sin\theta; \qquad h^{**} = -g^* \sin\theta + h^* \cos\theta$$

$$(2-\nu)g^{**} = -e^{i(\theta-\theta')}\cdot(G^*+G_0)_z + L_{0r} + (1-\nu)L_r^* + iL_\theta^*/r$$

$$(2-\nu)h^{**} = -ie^{i(\theta-\theta')}\cdot(G^*+G_0)_z + iL_{0r} - iL_r^* + (1-\nu)L_\theta^*/r$$

$$(2-\nu)\psi^* = -e^{i(\theta-\theta')}\cdot(G_r^*+G_{0r}+iG_\theta^*/r) - L_{0z} - (1-\nu)L_z^*$$

Mode III

$$(2-\nu)g^{**} = -i(1-\nu)e^{i(\theta-\theta')}\cdot(G^*+G_0)_z + i(1-\nu)(L_0-L^*)_r + L_\theta^*/r$$

$$(2-\nu)h^{**} = (1-\nu)e^{i(\theta-\theta')}\cdot(G^*+G_0)_z - (1-\nu)L_{0r} - L_r^* - i(1-\nu)L_\theta^*/r$$

$$(2-\nu)\psi^* = -i(1-\nu)[e^{i(\theta-\theta')}\cdot(G_r^*+G_{0r}+iG_\theta^*/r)+(L_0-L^*)_z].$$

6. SINGULARITY CONDENSATION

The fundamental fields which furnish the weight functions in (4.41), (5.20) and (5.21) will be considered in some detail. We assume (4.38) first and proceed with an analysis of

$$\tilde{G}^* = f_1(q)\sinh s + f_3(q)\sinh^3 s. \tag{6.1}$$

$\tilde{G}^*$ represents those terms in the expansion (4.39) of G^* which can cause unbounded stresses not only in the field of mode I for which G^* serves as Boussinesq–Papkovich potential, but also in the fields of the first and second kind, derived from G^*. From (4.14)

$$\sinh s = \frac{\alpha}{\sqrt{a}}(1+\rho/4a+0(\rho^2)), \qquad \sin t = \frac{\beta}{\sqrt{a}}(1-\rho/4a+0(\rho^2)) \tag{6.2}$$

(6.1) and (6.2) imply

$$\tilde{G}^* = f_1(q)\frac{\alpha}{\sqrt{a}}(1+\rho/4a)+f_3(q)\left(\frac{\alpha}{\sqrt{a}}\right)^3+0(\rho^{5/2}). \tag{6.3}$$

At this point an expansion of the functions of q is necessary. To this end we introduce two distances d, d' in the plane of the crack (Fig. 7). We define

$$d^2 = r^2+a^2-2ra\cos\vartheta; \qquad d'^2 = 2a^2(1-\cos\vartheta);\ \vartheta = \theta-\theta'. \tag{6.4}$$

We can write

$$f_1(q) = c_0\frac{q+1}{q-1} = c_0(r^2-a^2-2iar\sin\vartheta)/d^2. \tag{6.5}$$

Bearing in mind that $r-a = \rho\cos\varphi$ we observe that

$$\begin{aligned} r^2-a^2-2iar\sin\vartheta &= 2a(r-a)-2ia^2\sin\vartheta\left(1+\frac{r-a}{a}\right)+0(\rho^2) \\ d^2 = d'^2+2a(r-a)(1-\cos\vartheta)+0(\rho^2) &= d'^2\cdot\left(1+\frac{r-a}{a}\right)+0(\rho^2) \end{aligned} \tag{6.6}$$

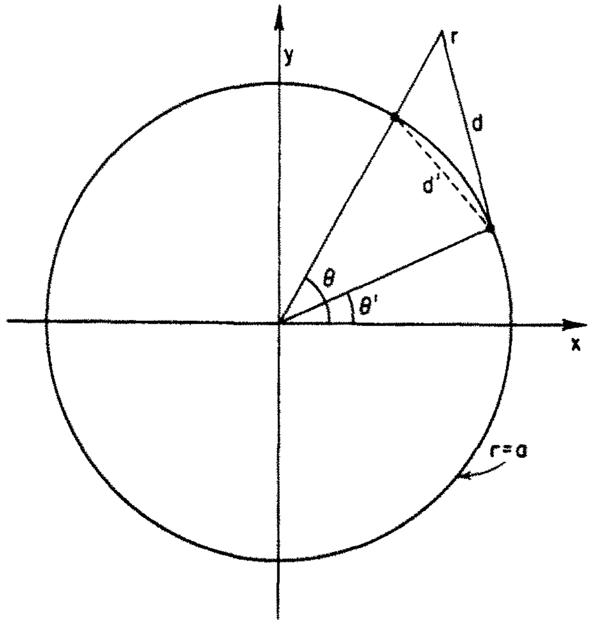


Fig. 7.

whence

$$1/d^2 = \left[1 - \frac{r-a}{a} + 0(\rho^2)\right] \Big/ d'^2$$
$$f_1(q) = 2c_0(a/d')^2 \cdot \left[-i \sin \vartheta + \frac{r-a}{a} + 0(\rho^2)\right]. \tag{6.7}$$

From (6.7)

$$3f_3 = 2c_0(a/d')^2 \cdot [i \sin \vartheta - 1 + 0(\rho)]. \tag{6.8}$$

Using (6.7) and (6.8) along with $\alpha^2 = \rho(1+\cos\varphi)$ we obtain

$$\tilde{G}^* = 2c_0(a/d')^2 \frac{\alpha}{\sqrt{a}} \cdot \{-i \sin \vartheta[1 + \rho/4a - \rho(1+\cos\varphi)/3a] + \rho(2\cos\varphi - 1)/3a\} + 0(\rho^{5/2}) \tag{6.9}$$

and in particular,

$$Re\ \tilde{G}^* = 2c_0(a/d')^2 \frac{\alpha}{\sqrt{a}} \cdot \frac{\rho}{a}(2\cos\varphi - 1) + 0(\rho^{5/2}). \tag{6.9a}$$

The displacements $Re\ W_1^*$ are generated by the Boussinesq–Papkovich potential $Re\ G^*$. Due to (6.9a) these displacements are bounded along any segment of the edge of the crack for which (4.38) holds. They also comply with (2.25); this makes the field by $Re\ G^*$ regular along those segments. The field is of mode I and possesses a stress intensity factor $k_1^*(\theta)$ for $\theta \neq \theta' \bmod \cdot 2\pi$. Here and in two more cases to come it is convenient to determine the stress intensity factors of an explicitly given field with the aid of (2.2) taken on C^+, i.e. for $\rho = a - r$ and $\varphi = \pi$. This yields

$$(1-\nu)k_1 = \lim_{\rho\to 0} \mu(w - w_0)/\sqrt{2\rho}; \qquad (1-\nu)k_2 = \lim_{\rho\to 0} \mu(u-u_0)/\sqrt{2\rho};$$
$$k_3 = -\lim_{\rho\to 0} \mu(v - v_0)/\sqrt{2\rho}. \tag{6.10}$$

In the particular case of W_I^* one finds (see also the list of crack face displacements in Section 7) an axial displacement component of $Re\ W_I^*$

$$w^* = \sqrt{2(a^2-r^2)}/\sqrt{a}d'^2 \qquad \text{on} \qquad C^+ \tag{6.11}$$

whence

$$k_I^*(\theta) = \frac{\sqrt{2}\mu}{(1-\nu)d'^2}. \tag{6.12}$$

[For simplicity the factor $\ell^{5/2}$, ℓ = unit of length, has been omitted on the right-hand sides of (6.11) and (6.12).] This stress intensity factor goes to infinity as $\theta \to \theta'$. It grows in inverse proportion to d'^2.

Let us now take a look at w^* for $\theta = \theta'$. In this case (6.11) yields

$$\theta = \theta' : w^* = \sqrt{\frac{2(a+r)}{a}}(a-r)^{-3/2}. \tag{6.11a}$$

We can now state: the summation to $Re\ W_I^*$ of the ordinary fundamental fields due to $Re\ G_n$ removes the unboundedness of the displacements at all points $\theta \neq \theta' \bmod 2\pi$ and aggravates the unboundedness at $\theta = \theta'$ to the order of $d^{-3/2}$ for some displacements.

The mode I field generated by $Im\ G^*$ is quite different. It is ordinary fundamental at all edge points $\theta \neq \theta' \bmod 2\pi$ with the geometric intensity coefficient

$$m_I^* = \frac{1}{a}\cot\tfrac{1}{2}(\theta-\theta'). \tag{6.13}$$

The coefficient goes to infinity as $\theta \to \theta'$ and changes sign after the passage through θ'.

The phenomena (6.11a), (6.12) and (6.13) suggest to call G^* a potential with a condensation point ($\theta = \theta'$) at the edge of the crack. The two associated fields, defined by $Re\ G^*$ and $Im\ G^*$, will be referred to as fundamental fields with a condensation point.

The phenomenon of condensation is also present in the fields of W_{II}^* and W_{III}^*. In order to exhibit the essential features we need the analogues of (6.9) with respect to G_0, L_0 and L^*. For $\rho \to 0$ we find

$$G_0 = c_0(\pi/2 - \sinh s + \tfrac{1}{3}\sinh^3 s) + 0(s^5). \tag{6.14}$$

The portion that can cause infinite stresses is

$$\tilde{G}_0 = -c_0 \sinh s + \frac{c_0}{3}\sinh^3 s = -c_0\frac{\alpha}{\sqrt{a}}[(1+\rho/4a - \rho(1+\cos\varphi)/3a] + 0(\rho^{5/2}) \tag{6.15}$$

(5.8) yields

$$L_0 = -c_0 \sin t + zG_0/a. \tag{6.16}$$

Unbounded stresses are due to

$$\tilde{L}_0 = -c_0\left(\sin t + \frac{z}{a}\sinh s\right) = -c_0\frac{\beta}{\sqrt{a}}[1-\rho/4a+\rho(1+\cos\varphi)/a] + 0(\rho^{5/2}). \tag{6.17}$$

As for L^* we go back to (5.17); using also (4.39) and (6.16), we find that

$$\tilde{L}^* = f_1(q)\sin t - 3\frac{z}{a}f_3(q)\sinh s \tag{6.18}$$

can be taken as that portion of L^* which gives rise to unbounded stresses. With the aid of (6.2), (6.7) and (6.8) we can write

$$\tilde{L}^* = 2c_0(a/d')^2\frac{\beta}{\sqrt{a}}\{-i\sin\vartheta[1+\rho(3+4\cos\varphi)/4a]+\rho(1+2\cos\varphi)/a\}+0(\rho^{5/2}). \tag{6.19}$$

The leading terms in the representations of $\tilde{G}_0$, $\tilde{L}_0$, $\tilde{G}^*$, $\tilde{L}^*$ are (in this order)

$$\hat{G}_0 = \alpha/\gamma, \quad \hat{L}_0 = \beta/\gamma, \quad \hat{G}^* = if(\vartheta)\alpha/\gamma, \quad \hat{L}^* = if(\vartheta)\beta/\gamma$$
$$\text{with} \quad \gamma = \sqrt{2}(1-\nu)a, \quad f(\vartheta) = \cot\tfrac{1}{2}\vartheta. \tag{6.20}$$

We must now determine to what extent these leading terms will prevail in $Re\ W^*_{\text{II}}$, $Re\ W^*_{\text{III}}$. The following list of displacements due to the leading terms is found:

$$\mathscr{D}_1(G^*+G_0)$$

$$\gamma e^{-i\theta'}u' = 2(1-\nu)(1-if)\alpha_z-(1-if)z\alpha_{rr}-(1-if+f')z(\alpha/r)_r$$
$$\gamma e^{-i\theta'}v' = 2(1-\nu)(i+f)\alpha_z+if'z\alpha_r/r-(f''-if')z\alpha/r^2$$
$$\gamma e^{-i\theta'}w = 2(1-\nu)[(1-if)\alpha_r+(1-if+f')\alpha/r]+[(-1+if)z\alpha_r-(1-if+f')z\alpha/r]_z$$

$$\mathscr{D}_2G^*;$$

$$\gamma u' = -2(1-\nu)[(1-\nu)if\beta_r-f'\beta/r]-i(1-\nu)fz\beta_{rz}$$
$$\gamma v' = -2(1-\nu)[f\beta_r+(1-\nu)if'\beta/r]-i(1-\nu)f'z\beta_z/r$$
$$\gamma w = 2i(1-\nu)^2f\beta_z-i(1-\nu)f(z\beta_z)_z$$

$$\mathscr{D}_2G_0;$$

$$\gamma u' = -2(1-\nu)^2\beta_r-(1-\nu)z\beta_{rz}$$
$$\gamma v' = 2(1-\nu)i\beta_r$$
$$\gamma w = 2(1-\nu)^2\beta_z-(1-\nu)(z\beta_z)_z.$$

These displacements show both bounded and unbounded terms. It turns out that the unbounded terms annihilate one another in the compositions $Re\ W^*_{\text{II}}$, $Re\ W^*_{\text{III}}$. We show this in detail for the component u' of $Re\ W^*_{\text{II}}$. By (5.18a) and the preceding list

$$\begin{aligned}(2-\nu)\gamma u' &= -2(1-\nu)\beta_r-z\beta_{rz}+2(1-\nu)f'\beta/r-2(1-\nu)\alpha_z+z\alpha_{rr}+(1+f')z(\alpha/r)_r\\ &= 2(1-\nu)f'\beta/r+(1+f')z(\alpha_r/r-\alpha/r^2),\end{aligned} \tag{6.21}$$

since $\alpha_r = \beta_z$ and $\alpha_z = -\beta_r$. The terms in the last line are bounded and they vanish at the edge. We emphasize here that (6.21) represents only the contribution of the leading terms (6.20). In order to obtain the complete u' we must add the contribution from $\mathscr{D}_1(G^*-\hat{G}^*+G_0-\hat{G}_0)$, $\mathscr{D}_2(G^*-\hat{G}^*)$, $\mathscr{D}_2(G_0-\hat{G}_0)$. Obviously these yield bounded displacements so that the total u' for $Re\ W^*_{\text{II}}$ is bounded. Moreover it abides by (2.25); the last line in (6.21) illustrates this phenomenon. To show the same for the other components of $Re\ W^*_{\text{II}}$ and for all of $Re\ W^*_{\text{III}}$ is left to the reader.

Altogether we can state that the fields of $Re\ W^*_{\text{II}}$, $Re\ W^*_{\text{III}}$ behave like regular ones for

$\theta \neq \theta' \bmod \cdot 2\pi$. As such, they exhibit stress intensity factors. We state that

$$k_1^* = 0, \qquad k_2^* = \frac{\sqrt{2}\mu}{(1-\nu)(2-\nu)}[(2-3\nu)/d'^2 - 1/a^2]$$

$$k_3^* = \frac{2\sqrt{2}\mu}{(2-\nu)} \cdot \sin\vartheta/d'^2$$

$$\text{for mode II.} \tag{6.22}$$

$$k_1^* = 0, \qquad k_2^* = -\frac{2\sqrt{2}\mu}{(2-\nu)}\sin\vartheta/d'^2$$

$$k_3^* = \frac{\sqrt{2}\mu}{2-\nu}[(2+\nu)/d'^2 - (1-\nu)/a^2]$$

$$\text{for mode III.} \tag{6.23}$$

Again (6.10) can be used in order to derive (6.22) and (6.23) from asymptotic or explicit representations of the displacements on C^+. In particular the list of crack face weight functions in the next section can be chosen. In this context one should observe that

$$2\lambda + \vartheta = \pm\pi \qquad \text{for} \qquad r = a. \tag{6.24}$$

With this much said, the detailed verification of (6.22) and (6.23) can be left to the interested reader. The behavior of k_2^* for mode II and of k_3^* for mode III is similar to that of k_1^* for mode I. It is interesting to note that k_3^* for mode II and k_2^* for mode III do not vanish. They also go to infinity as $\theta \to \theta'$; their order of growth is given by $2a^2 \sin\vartheta/d'^2 = \cot\frac{1}{2}\vartheta$, and they change sign as θ passes through θ'. We shall return to these phenomena in our discussion of the half-plane crack.

Relation (4.28) of the reciprocity theorem can be replaced by

$$\int_\omega [(W, \mathbf{T}^*) - (W^*, \mathbf{T})]\, dS = \int_{S'} [(W^*, \mathbf{T}) - (W, \mathbf{T}^*)]\, dS + \int_{V'} (W^*, \mathbf{F})\, dV,$$

where W^* is either $Re\, W_{\mathrm{I}}^*$, $Re\, W_{\mathrm{II}}^*$ or $Re\, W_{\mathrm{III}}^*$ and $\mathbf{T}^*$ is the associated traction vector. A comparison between formulas (4.41), (5.20) and (5.21) shows that:

$$\lim_{\rho \to 0} \int_\omega [(W, \mathbf{T}^*) - (W^*, \mathbf{T})]\, dS = 2\sqrt{2} \cdot \pi^2 \cdot k_j(\theta'), \tag{6.25}$$

where $j = 1, 2, 3$ in accord with the mode chosen for W^*. Now the surface of the torus ω in (6.25) can be split into pieces ω', ω'' with ω'' satisfying (4.38). Since the field of W^* is regular under (4.38) we have

$$\lim_{\rho \to 0} \int_{\omega''} [(W, \mathbf{T}^*) - (W^*, \mathbf{T})]\, dS = 0. \tag{6.26}$$

Combining (6.25) and (6.26) we see that

$$\lim_{\rho \to 0} \int_{\omega'} [(W, \mathbf{T}^*) - (W^*, \mathbf{T})]\, dS = 2\sqrt{2}\pi^2 k_j(\theta') \tag{6.27}$$

no matter how small $\varepsilon > 0$. This relation applies to any regular field with intensity factors

k_j at $\theta = \theta'$. Therefore, the fundamental fields associated with $Re\ W_{\rm I}^*$, $Re\ W_{\rm II}^*$ and $Re\ W_{\rm III}^*$ have universal character near the point of condensation.

7. THE HALF-PLANE CRACK. A LIST OF CRACK FACE WEIGHT FUNCTIONS

We return to the configurations Figs 4 and 5. One can derive the weight function formulas (4.41), (5.20) and (5.21) for the half-plane crack by a suitable adaptation of the procedures used for the penny-shaped crack. The equivalent of the potentials G_n is[4]

$$G(x,y,z) = \text{constant}\cdot Erfc(\sqrt{\lambda}\alpha)e^{\lambda(x+iy)}, \qquad \lambda > 0. \tag{7.1}$$

The series (4.33) is to be replaced by a Fourier integral. The potential L is derived from G by means of (3.20).

We can also obtain the weight functions for the half-plane crack by a limit procedure. To this end we place the center of the penny-shaped crack of radius a at the point $x = -a$, $y = 0$. As $a \to \infty$ the penny-shaped crack turns into the half-plane $x < 0$. Following this procedure we confine the point x, y and z to a fixed finite domain. The coordinate of condensation on the crack edge is denoted by y'. It takes the place of θ'. We have to use

$$r = a+x+0(1/a), \qquad \theta = y/a+0(a^{-2}) \qquad \text{as} \qquad a \to \infty \tag{7.2}$$

whence

$$\begin{gathered} a(q-1) \sim x+i(y-y') = \zeta, \qquad 2a\sinh^2 \tfrac{1}{2}(s+it) = r-a+iz \sim x+iz = \rho e^{i\varphi} \\ s+it \sim \frac{1}{\sqrt{a}}(\alpha+i\beta). \end{gathered} \tag{7.3}$$

The potential G^*, defined by (4.35), goes to the limit

$$G_* = \frac{1}{\sqrt{2}\cdot(1-\nu)\sqrt{\zeta}} \log\frac{\alpha-\sqrt{\zeta}}{\alpha+\sqrt{\zeta}}. \tag{7.4}$$

This result is partially due to the influence of c_0 since this coefficient is in proportion to $a^{-1/2}$. We observe that $G_0 \to 0$ as $a \to \infty$. For fixed $G(x,y,z)$ the function L of (3.22) depends on a. As $a \to \infty$, the function L takes the limit (3.20). In the case $G = G^*$ the limit of L^* as $a \to \infty$ turns out to be

$$L_* = 2(zG_{*x} - xG_{*z}), \tag{7.5}$$

i.e. L_* is found by direct application of (3.20) to G_*. The weight function formulas retain their form after replacing θ' by y'. The displacements $Re\ W_{\rm I}^*$, etc. are obtained as follows:

For mode I. Use $Re\ G_*$ as Boussinesq–Papkovich potential and determine the displacements by means of (3.2).

For modes II and III. Determine displacements of the first kind by (3.12) and (3.15) for $G = G_*$; find the displacements of the second kind by (3.12) and (3.17) for $L = L_*$. The displacement vectors so obtained are combined into

$$(2-\nu)W_{\rm II}^* = \mathscr{D}_2 G_* - \mathscr{D}_1 G_* \tag{7.6}$$

$$(2-\nu)W_{\rm III}^* = -i\mathscr{D}_2 G_* - i(1-\nu)\mathscr{D}_1 G_*. \tag{7.7}$$

From these it follows that

$$(2-\nu)Re\ W_{\rm II}^* = Re\,(\mathscr{D}_2 G_* - \mathscr{D}_1 G_*) \tag{7.8}$$

$$(2-\nu)Re\, W^*_{III} = Im\,[\mathscr{D}_2 G_* + (1-\nu)\mathscr{D}_1 G_*]. \tag{7.9}$$

With these rules and formulas the weight functions for the half-plane crack are completely described.

In many applications the weight functions for the crack faces C^+, C^- alone are needed. For the convenience of the reader we list the displacements of the relevant fundamental fields for penny and half-plane. It suffices to do this for C^+. The displacements on C^- are found by the following rules: the normal displacements w of modes II and III and the displacements u' and v' (u and v) of mode I are the same at opposite points of the crack. All others change sign as we go from a point on one face to the opposite one of the other.

Crack face weight functions on penny

Abbreviations: $\nu' = 2(1-\nu)/(2-\nu)$; $e^{i\vartheta}\cdot(G_z^* + G_{0z}) = G'$

Preliminary Forms:

Mode I.

$$u' = -(1-2\nu)Re\, G_r^*, \qquad v' = -(1-2\nu)Re\, G_\theta^*/r, \qquad w = 2(1-\nu)Re\, G_z^*.$$

Mode II.

$$u' = -\nu' Re\,[L_{0r} + (1-\nu)L_r^* + iL_\theta^*/r - G']$$

$$v' = \nu' Re\,[iL_r^* - (1-\nu)L_\theta^*/r + iG'], \qquad w = \nu' Re\,[L_{0z} + (\tfrac{1}{2}-\nu)L_z^*].$$

Mode III.

$$u' = \nu' Re\,[(1-\nu)iL_r^* - L_\theta^*/r + i(1-\nu)G']$$

$$v' = \nu' Re\,[(1-\nu)L_{0r} + L_r^* + i(1-\nu)L_\theta^*/r - (1-\nu)G']$$

$$w = \nu'(\tfrac{1}{2}-\nu)\, Im\, L_z^*.$$

With the abbreviation $T = \sqrt{a^2-r^2}$ the quantities in the preliminary forms take these values on C^+:

$$G_r^* = -\tfrac{1}{2}c_0\pi i\frac{q}{r(q-1)\sqrt{q-1}}, \qquad G_\theta^*/r = iG_r^*, \qquad G_z^* = c_0\frac{q+1}{(q-1)T}$$

$$G' = 2c_0 e^{i\vartheta}/(q-1)T, \qquad L_{0r} = c_0 r/aT, \qquad L_{0z} = c_0\pi/2a$$

$$L_r^* = -2c_0\frac{qT}{ar(q-1)^2} - c_0\frac{(q+1)r}{(q-1)aT}; \qquad L_\theta^*/r = -2ic_0\frac{qT}{ar(q-1)^2}$$

$$L_z^* = -c_0\pi/2a - c_0\pi i\frac{1}{a(q-1)\sqrt{q-1}}.$$

In order to express real and/or imaginary parts of these quantities in a convenient way, we use besides r, d, ϑ, the angles δ and λ [no relation to the parameter λ in (7.1)] as shown in Fig. 8. We can write

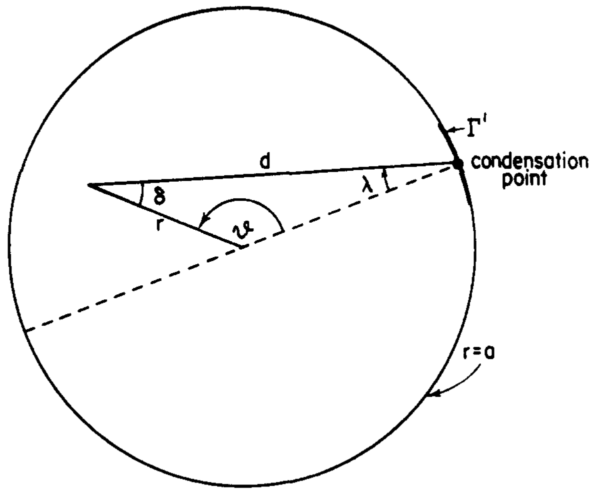
$$\frac{1}{q-1} = -ae^{i\lambda}/d, \qquad \frac{q+1}{q-1} = 1 - 2ae^{i\lambda}/d$$

$$\frac{q}{(q-1)^2} = -ae^{i\lambda}/d + (a/d)^2 e^{2i\lambda}; \qquad \frac{1}{\sqrt{q-1}} = \sqrt{a/d}\,(\sin\lambda/2 - i\cos\lambda/2)$$

$$\frac{1}{(q-1)\sqrt{q-1}} = (a/d)^{3/2}\cdot(-\sin 3\lambda/2 + i\cos 3\lambda/2);$$

$$\frac{q}{(q-1)\sqrt{q-1}} = \frac{1}{\sqrt{q-1}} + \frac{1}{(q-1)\sqrt{q-1}}$$

$$G' = -2c_0(a/d\cdot T)e^{i(\lambda+\vartheta)} = 2c_0(a/dT)e^{-i\delta}.$$



sgn λ = sgn ρ = sgn ϑ for 0<|ϑ|<π

Fig. 8.

The following two identities are useful in order to obtain the final form of the displacements:

$$[2(a/d)\cos\delta - r/a]/T = T[1-(a/d)^2]/ar \tag{7.10}$$

$$[(2a^2/rd)\cos\lambda - r/a]/T = T[1+(a/d)^2]/ar. \tag{7.11}$$

Final Forms:

Mode I.

$$u' = \frac{(1-2\nu)\pi}{(1-\nu)2r\sqrt{2d}}[-\cos\lambda/2 + (a/d)\cos 3\lambda/2]$$

$$v' = \frac{(1-2\nu)\pi}{(1-\nu)\cdot 2r\sqrt{2d}}[\sin\lambda/2 - (a/d)\sin 3\lambda/2]$$

$$w = \sqrt{2}T/\sqrt{a}d^2.$$

Mode II.

$$u' = -\frac{\sqrt{2}\cdot T}{(2-\nu)\cdot\sqrt{aar}}\{\nu + 2(a/d)\cos\lambda + (a/d)^2[\nu - 2 - 2\nu\cdot\cos 2\lambda]\}$$

$$v' = -\frac{2\sqrt{2}\cdot T}{(2-\nu)\cdot\sqrt{aar}}[(1-\nu)(a/d)\sin\lambda + \nu(a/d)^2\sin 2\lambda]$$

$$w = -\frac{\sqrt{2}\cdot\pi}{2(2-\nu)a\cdot\sqrt{a}}[\tfrac{1}{2} + \nu + (1-2\nu)(a/d)^{3/2}\cos 3\lambda/2].$$

Mode III.

$$u' = \frac{2\sqrt{2}\cdot T}{(2-\nu)\cdot\sqrt{aar}}[-(a/d)\sin\lambda + \nu(a/d)^2\sin 2\lambda]$$

$$v' = -\frac{\sqrt{2}\cdot T}{(2-\nu)\cdot\sqrt{aar}}[\nu - 2(1-\nu)(a/d)\cos\lambda + (a/d)^2(2-\nu-2\nu\cos 2\lambda)]$$

$$w = -\frac{(1-2\nu)\sqrt{2}\pi}{2(2-\nu)\cdot d^{3/2}}\sin 3\lambda/2.$$

At $r = 0$ the displacements u', v' become ambiguous and Cartesian displacements u, v are preferable. The transformation from one pair to the other is—in complex form—

$$u+iv = e^{i\theta}(u'+iv'). \tag{7.12}$$

As an example let us consider mode I. From the list of final forms we derive

$$u+iv = \frac{(1-2\nu)}{(1-\nu)2r\sqrt{2d}} \cdot e^{-i\lambda/2} \cdot e^{i\theta}[-1+(a/d)e^{-i\lambda}];$$

here

$$e^{i\theta}(-1+(a/d)e^{-i\lambda}) = e^{i\theta} \cdot \frac{\bar{q}}{1-\bar{q}} = e^{i\theta'} \frac{r}{a(1-\bar{q})}.$$

As $r \to 0$ we have $q \to 0$, $\lambda \to 0$ and

$$u+iv = \frac{(1-2\nu)\pi}{(1-\nu)\cdot 2a\sqrt{2a}} \cdot e^{i\theta'}.$$

The same result is found as we use the preliminary list and

$$G_r^* \to \frac{1}{2a} \cdot c_0 \pi e^{i\vartheta} \qquad \text{as} \qquad r \to 0 \qquad \text{at constant } \vartheta$$

together with $G_\theta^*/r = iG_r^*$. The other modes can-be treated in the same vein.

The weight functions for the half-plane crack, in particular on C^+, C^-, can be derived from those of the penny by letting $a \to \infty$. The quantities d, d', λ retain their previous meaning, and Fig. 9 shows the relevant details. We observe that

$$T/\sqrt{2a} \to \sqrt{-x} \qquad \text{as } a \to \infty. \tag{7.13}$$

In the limit process the term ar behaves like a^2. The displacements u', v' become Cartesian ones u, v. We list the final results.

Crack face weight functions on half-plane C^+

Mode I.

$$u = \frac{(1-2\nu)\sqrt{2\pi}}{4(1-\nu)} d^{-3/2} \cdot \cos 3\lambda/2$$

$$v = \frac{-(1-2\nu)\cdot\sqrt{2\pi}}{4(1-\nu)} d^{-3/2} \sin 3\lambda/2$$

$$w = 2\cdot\sqrt{-x}/d^2.$$

Mode II.

$$u = 2\left(1+\frac{2\nu}{2-\nu}\cos 2\lambda\right)\cdot\sqrt{-x}/d^2$$

$$v = -\frac{4\nu}{2-\nu}(\sin 2\lambda)\sqrt{-x}/d^2$$

$$w = \frac{-\sqrt{2}(1-2\nu)\pi}{2(2-\nu)} d^{-3/2} \cos 3\lambda/2.$$

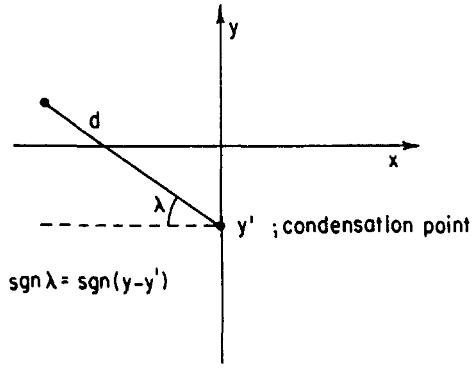


Fig. 9.

Mode III.

$$u = \frac{4\nu}{(2-\nu)}(\sin 2\lambda)\cdot\sqrt{-x}/d^2$$

$$v = -2\left(1-\frac{2\nu}{2-\nu}\cos 2\lambda\right)\sqrt{-x}/d^2$$

$$w = -\frac{\sqrt{2}\cdot(1-2\nu)\pi}{2-\nu}d^{-3/2}\sin 3\lambda/2.$$

The limit process $a \to \infty$ can also be applied to the formulas (6.12), (6.22) and (6.23) for the stress intensity factors of the fundamental fields at the edge points $\theta \neq \theta' \bmod \cdot 2\pi$. Relation (6.12) stays verbally intact; (6.22) and (6.23) become simpler. For the half-plane crack k_2^* vanishes for mode III and k_3^* for mode II. The terms $1/a^2$ are to be replaced by zero. That k_2^* does not vanish in (6.23) and that k_3^* does not vanish in (6.22) must be interpreted as the effect of curvature of the edge of the crack.

We conclude this section with two examples. For our first example we expose the crack faces of the penny to distributed pressure $p(r,\theta)$. Formula (4.41) and the displacement $w = \sqrt{2}T/\sqrt{a}d^2$ of mode I lead immediately to (1.9). In an earlier derivation of (1.9) in [4] the displacement w was found by summation of (4.32) with the W_n^* taken at the crack faces. The second example is for the half-plane crack. At the point $x = -d_0$, $y = 0$ of the crack we apply forces Y to C^+, $-Y$ to C^-; the force vectors are parallel to the y-axis (the edge of the crack). In this case the elastic deformation is of mixed modes (II and III). Equations (5.20) and (5.21) are to be used in conjunction with the displacement v. In the list above, v is given with the aid of λ. We have

$$d_0+iy' = d\cdot e^{-i\lambda},\qquad d^2 = d_0^2+y'^2,\qquad d^2\cos 2\lambda = d_0^2-y'^2,\qquad d^2\sin 2\lambda = -2d_0y'.$$

This leads to the stress intensity factors:

$$k_2(y') = \frac{4\nu Y}{(2-\nu)\pi^2}\sqrt{2d_0}\cdot d_0y'/d^4 \tag{7.14}$$

$$k_3(y') = -\frac{Y}{\pi^2d^2}\sqrt{2d_0}\left(1-\frac{2\nu}{2-\nu}\cdot\frac{d_0^2-y'^2}{d^2}\right). \tag{7.15}$$

8. FINAL REMARKS

For the penny-shaped and the half-plane crack in the full space all relevant fundamental fields are now available. The associated weight functions permit the calculation of the stress intensity factors of interest by mere quadrature but our results are significant beyond such applications.

Let us consider a finite elastic body $\mathscr{B}$ with a penny-shaped crack inside, say the body in Fig. 6, bounded by the spherical surface Ω. Let $\mathscr{F}'$ denote one of the fundamental fields with condensation point θ' of the infinite structure, say the field of mode I, generated by $Re\ G^*$. One can show that the tractions on Ω, generated by $\mathscr{F}'$, are self-equilibrated. It is therefore possible to determine within $\mathscr{B}$ a regular field $\mathscr{R}'$ which annihilates those tractions. Moreover $\mathscr{R}'$ can be assumed to have no body forces. The sum $\mathscr{F} = \mathscr{F}'+\mathscr{R}'$ constitutes a field in $\mathscr{B}$ without body forces and without tractions on C^+, C^- and Ω. It is a fundamental field for $\mathscr{B}$. Assume now in $\mathscr{B}$ a regular field $\mathscr{R}$, whose stress intensity factors we wish to determine. To this end we apply the reciprocity theorem to $\mathscr{R}$ and $\mathscr{F}$ for that portion $\mathscr{B}'$ of $\mathscr{B}$ which is outside the torus ω (Fig. 6). If W and W^* denote the displacement vectors of $\mathscr{R}$ and $\mathscr{F}$ respectively, if furthermore $\mathbf{T}$, $\mathbf{F}$ are traction and body force of $\mathscr{R}$ while $\mathbf{T}^*$ stands for the traction due to $\mathscr{F}$, then

$$\int_{\omega} [(W,\mathbf{T}^*)-(W^*,\mathbf{T})]\,\mathrm{d}S = \int_{S'} (W^*,\mathbf{T})\,\mathrm{d}S+\int_{\mathscr{B}'}(W^*,\mathbf{F})\,\mathrm{d}V;$$

S' consists of Ω and of those portions of C^+, C^-, which are outside the torus. As $\rho' \to 0$ this equation takes the form

$$2\cdot\sqrt{2}\pi^2 k_1^*(\theta') = \int_{C\cup\Omega} (W^*,\mathbf{T})\,\mathrm{d}S+\int_{\mathscr{B}}(\mathbf{W}^*,\mathbf{F})\,\mathrm{d}V. \tag{8.1}$$

The right-hand side needs no explanation. The left-hand side is based on the observations (6.25)–(6.27) with regard to $\mathscr{F}'$ and on the circumstance that the contribution of $\mathscr{R}'$ to the integral over ω will vanish in the limit $\rho' \to 0$. Formula (8.1) can be extended towards k_2 and k_3. Moreover the validity of (8.1) is not confined to a body bounded by a sphere. Ω can be any surface containing the crack. The computational effort to acquire a weight function formula of type (8.1) is modest. We merely have to modify an already available fundamental field by a regular field $\mathscr{R}'$.

We can go further. Let $\mathscr{B}$ be a finite body with a plane crack of convex shape. Moreover we assume the edge of the crack to be smooth and to be composed of circular arcs of various curvatures. Let Γ' be such an arc. Without loss of generality we can assume that Γ' lies on a circle of radius a, as shown in Fig. 8. We now pick a point of condensation on Γ' (θ' in Fig. 8) and draw a sphere of radius ε (called an ε sphere) around it. For sufficiently small values of ε the arc will have its endpoints outside the sphere; all points on and within that sphere will belong to $\mathscr{B}$. We take again the fundamental field $\mathscr{F}'$ of mode I and condensation point θ', as generated by $Re\ G^*$ for the penny-shaped crack in the infinite space. We define: a field $\mathscr{F}$ in $\mathscr{B}$ is fundamental of mode I with condensation point θ' if: (a) the field has neither body forces nor tractions on the crack and other boundary of $\mathscr{B}$; (b) it behaves like a regular one at all points of the edge, different from the point of condensation; (c) within the ε sphere it differs from $\mathscr{F}'$ by a regular field.

For such a field $\mathscr{F}$ and any regular field in $\mathscr{B}$, formula (8.1) applies again; this time C is the plane crack of convex shape and Ω the remaining boundary of $\mathscr{B}$. In the same vein, fundamental fields with condensation point θ' can be defined for the other modes. In [10], Paris, McMeeking and Tada described a numerical method for the calculation of plane strain fundamental fields. It is based on *a priori* knowledge of the asymptotic behavior of the displacements of the fundamental field near the tip of the crack. Their method can be utilized for the computation of $\mathscr{F}$. Inside the ε sphere we approximate $\mathscr{F}$ by $\mathscr{F}'$; in the remaining portion $\mathscr{B}_0$ of $\mathscr{B}$ we approximate $\mathscr{F}$ by a regular field $\mathscr{R}_0$ without body force, subject to these conditions: (i) on the ε sphere, the displacements of $\mathscr{R}_0$ and $\mathscr{F}'$ coincide; (ii) $\mathscr{R}_0$ has no tractions on all other parts of the boundary of $\mathscr{B}_0$.

On this occasion, some comments on the uniqueness of weight functions are perhaps in order. Returning to (1.1) we assume that W^* is continuously defined throughout the elastic region, i.e. at all points of V other than those on the edge of the crack. W^* shall be continuously differentiable at the inner points of the region. As for $W^{*\prime}$, this vector field shall be continuous at all points of the boundary S of V, the edge excluded. Let there be a second pair of such fields, $\tilde{W}^*$ and $\tilde{W}^{*\prime}$ for V and S respectively such that

$$k_j(Q) = \int_V (\mathbf{F}, \tilde{W}^*)\, \mathrm{d}V + \int_S (\mathbf{T}, \tilde{W}^{*\prime})\, \mathrm{d}S, \tag{8.2}$$

for all regular fields responding to self-equilibrated load systems **F**, **T**. Subtracting (8.2) from (1.1) we find

$$\begin{gathered} 0 = \int_V (\mathbf{F}, W_0)\, \mathrm{d}V + \int_S (\mathbf{T}, W_0')\, \mathrm{d}S \\ \text{with} \quad W_0 = W^* - \tilde{W}^*;\; W_0' = W^{*\prime} - \tilde{W}^{*\prime}. \end{gathered} \tag{8.3}$$

If W_0 is a displacement field of rigid body motion and if $W_0' = W_0$ on S then (8.3) is satisfied for all self-equilibrated load systems **F**, **T**. We assert that this is the only choice of W_0, W_0' in order to make (8.3) possible. Proof: let P_1, P_2 denote position vectors of two inner points of V and let concentrated body forces attack P_1 and P_2, such that $\lambda(P_1 - P_2)$ is the force at P_1 and $\lambda(P_2 - P_1)$ the force at P_2, λ being some scalar. This system of forces is self-equilibrated. From (8.3) it follows that the component of W_0 in the direction of $P_2 - P_1$ is the same at both points and, more generally, the same at all points $P = P_1 + t(P_2 - P_1)$ where t is a scalar restricted by the condition that P belong to V. Since W_0 is continuously differentiable inside V, one can derive that W_0, interpreted as a displacement field, has vanishing strains. Consequently W_0 is a displacement field of rigid body motion. With this taken into account, (8.3) implies

$$0 = \int_S (\mathbf{T}, W_0' - W_0)\, \mathrm{d}S \tag{8.4}$$

for all self-equilibrated systems **F**, **T**, but **F** does not appear in (8.4) and (8.4) can be taken as valid for arbitrary **T**, which in turn implies $W_0' = W_0$, Q.E.D.

From here on it is reasonable to assume that the weight functions W^*, $W^{*\prime}$ in (1.1) are displacements of a suitably normalized fundamental field $\mathscr{F}$ with condensation point Q. The stress intensity factor $k_j(Q)$ in (1.1) is that of a regular field, responding to body forces **F** and boundary tractions **T**. Let us now change the causes which give rise to the regular field. Let the displacements of such a field ($\mathscr{R}$) be prescribed on a portion S_0 of S (Fig. 1, shaded area) and let the tractions be given on the remaining portion S_1 of S. Body forces are admitted as before. In order to find a formula for $k_j(Q)$ under the new circumstances we modify $\mathscr{F}$ by the addition of a regular field $\mathscr{R}_0$ such that the sum $\tilde{\mathscr{F}} = \mathscr{F} + \mathscr{R}_0$ yields zero displacements on S_0. In this context it is assumed that S_0 is away from the edge of the crack. Altogether $\tilde{\mathscr{F}}$ is a field without body forces, without tractions on S_1 and without displacements on S_0. With $\mathscr{R}$ and $\tilde{\mathscr{F}}$ we form the energy balance on the reciprocity theorem and obtain

$$k_j(Q) = \int_{S_1} (\mathbf{T}, W^*)\, \mathrm{d}S - \int_{S_0} (\mathbf{T}^*, W)\, \mathrm{d}S + \int_V (\mathbf{F}, W^*)\, \mathrm{d}V \tag{8.5}$$

where W^*, $\mathbf{T}^*$ go with $\tilde{\mathscr{F}}$. This brings us to another utilization of the fundamental fields of the penny-shaped and of the half-plane crack. If a finite body with a penny-shaped crack inside is subject to geometric boundary conditions one can acquire weight function formulas of type (8.5) by merely adding regular fields to the fundamental ones, described in the preceding sections.

The infinite structure, cracked in a plane outside of a circle of radius a, can also be treated by the method of crack-analytic potentials and with the aid of relation (3.22) between G and L. Its fundamental fields, modified by the addition of regular ones, can be useful in the analysis of finite structures such as the cylinder with an external circumferential notch.

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APPENDIX A: HARMONICITY-PRESERVING LINEAR DIFFERENTIAL OPERATORS OF ORDER 1

We take a domain D (an open and connected set) of the space and consider the class $\mathcal{G}$ of functions which are harmonic in D. If $\mathbf{F}(x, y, z)$ belongs to $\mathcal{G}$ then $\mathbf{F}$ admits continuous partial derivatives of all orders and satisfies

$$\nabla^2\mathbf{F} = \mathbf{F}_{xx}+\mathbf{F}_{yy}+\mathbf{F}_{zz} = 0. \tag{A1}$$

Let P, X, Y, Z be four functions, defined in D, each of them having continuous partial derivatives up to the second order. We construct the linear operator

$$\mathcal{L} = P+X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+Z\frac{\partial}{\partial z} \tag{A2}$$

and pose this question: how must P, X, Y, Z be chosen in order to enforce

$$\mathcal{L}\mathbf{F}\in G \qquad \text{for all} \qquad \mathbf{F}\in\mathcal{G}. \tag{A3}$$

Applying $\mathcal{L}$ to the harmonic functions $\mathbf{F} = 1$, $\mathbf{F} = x$, $\mathbf{F} = y$, $\mathbf{F} = z$ we see at once that

$$P\in\mathcal{G} \tag{A4}$$

$$X+x\cdot P\in\mathcal{G}, \qquad Y+y\cdot P\in\mathcal{G}, \qquad Z+z\cdot P\in\mathcal{G} \tag{A5}$$

are necessary conditions in order to achieve (A3). They imply in particular that P, X, Y, Z must admit continuous partial derivatives of any order. Assuming (A4), (A5) from here on and using

$$\nabla^2 P = 0 \tag{A6}$$

$$\nabla^2 X + 2P_x = 0, \qquad \nabla^2 Y + 2P_y = 0, \qquad \nabla^2 Z + 2P_z = 0 \tag{A7}$$

we obtain

$$\frac{1}{2}\cdot\nabla^2\mathscr{L}\mathbf{F} = X_x\mathbf{F}_{xx} + Y_y\mathbf{F}_{yy} + Z_z\mathbf{F}_{zz} + (X_y + Y_x)\mathbf{F}_{xy} + (X_z + Z_x)\mathbf{F}_{xz} + (Y_z + Z_y)\mathbf{F}_{yz} \tag{A8}$$

for a generic $\mathbf{F}\in\mathscr{G}$. For the special harmonic functions

$$\mathbf{F} = xy, \qquad \mathbf{F} = xz, \qquad \mathbf{F} = yz, \qquad \mathbf{F} = x^2 - y^2, \qquad \mathbf{F} = x^2 - z^2, \qquad \mathbf{F} = y^2 - z^2$$

the condition $\nabla^2\mathscr{L}\mathbf{F} = 0$ and (A8) necessitate

$$X_y + Y_x = 0, \qquad X_z + Z_x = 0, \qquad Y_z + Z_y = 0 \tag{A9}$$

$$X_x = Y_y = Z_z. \tag{A10}$$

Now (A8), (A9) and (A10) imply

$$\nabla^2\mathscr{L}\mathbf{F} = 2X_x\nabla^2\mathbf{F} = 0 \tag{A11}$$

for a generic $\mathbf{F}\in\mathscr{G}$. Altogether we can now state that conditions (A4), (A5), (A9) and (A10) are necessary and sufficient to enforce (A3), i.e. to make $\mathscr{L}\mathbf{F}$ harmonic whenever $\mathbf{F}$ is so. It is easily checked that the following functions comply with those conditions:

$$P = P_0 + ax + by + cz \tag{A12}$$

$$X = a(x^2 - y^2 - z^2) + 2bxy + 2cxz + dx - c^*y + b^*z + X_0 \tag{A13}$$

$$Y = 2axy + b(y^2 - x^2 - z^2) + 2cyz + dy + c^*x - a^*z + Y_0 \tag{A14}$$

$$Z = 2axz + 2byz + c(z^2 - x^2 - y^2) + dz - b^*x + a^*y + Z_0 \tag{A15}$$

where P_0, X_0, Y_0, Z_0; a, b, c, d; a^*, b^*, c^* are constant coefficients. The case where all coefficients but P_0, X_0, Y_0, Z_0 vanish can be considered as trivial. Of the others we list three in particular. These are

$$\mathscr{L}\mathbf{F} = x\mathbf{F}_x + y\mathbf{F}_y + z\mathbf{F}_z \tag{A16}$$

$$\mathscr{L}\mathbf{F} = z\mathbf{F}_x - x\mathbf{F}_z \tag{A17}$$

$$\mathscr{L}\mathbf{F} = z\mathbf{F} + 2z(x\mathbf{F}_x + y\mathbf{F}_y) + (z^2 - x^2 - y^2)\mathbf{F}_z. \tag{A18}$$

The operators of (A17) and (A18) are useful in the crack analysis of this paper. In terms of cylindrical coordinates r, θ, z we can represent (A18) in the form

$$\mathscr{L}\mathbf{F} = z\mathbf{F} + 2zr\mathbf{F}_r + (z^2 - r^2)\mathbf{F}_z. \tag{A19}$$

The operators defined by (A12)–(A15) are the only ones complying with (A4), (A5), (A9) and (A10). We consider the function P. From (A7) and (A9) we derive $(X_y + Y_x) = -4P_{xy} = 0$; in the same vein $P_{xz} = 0$, $P_{yz} = 0$ follow. (A7) and (A10) yield

$$\nabla^2 X_x = -2P_{xx} = \nabla^2 Y_y = -2P_{yy} = \nabla^2 Z_z = -2P_{zz};$$

this and the harmonicity of P imply $P_{xx} = P_{yy} = P_{zz} = 0$. We have shown that all second order derivatives of P vanish; (A12) follows. Turning to X, Y, Z we derive from (A9) and (A10): $X_{xx} = Y_{yx} = -X_{yy}$, $X_{xx} = -X_{zz}$, hence $X_{xx} = -\nabla^2 X = 2P_x$. Integration yields $X_x = 2P + u(y, z)$; analogously $Y_y = 2P + v(x, z)$, $Z_z = 2P + w(x, y)$. Due to (A10) we must have $u = v = w$, which cannot happen unless u, v and w are constants. Altogether we have found that

$$X_x = 2P + d, \qquad Y_y = 2P + d, \qquad Z_z = 2P + d \tag{A20}$$

where d is a constant. At this juncture we derive from (A9):

$$X_{yz} = -Y_{xz} = Z_{xy} = -X_{yz} \qquad \text{whence} \qquad X_{yz} = Y_{xz} = Z_{xy} = 0. \tag{A21}$$

Let now d and P be given, the latter in the form (A12). It remains to determine X, Y and Z from (A20) by integration, complying with conditions (A9). It is obvious that X, Y and Z by (A13), (A14) and (A15), satisfy (A20) as well as (A9); in particular (A20) and (A9) are already satisfied by those terms of X, Y, Z which go with the coefficients a, b, c, d. The remaining portions of X, Y and Z satisfy (A9) and the homogeneous forms of

(A20), i.e. the equations

$$X_x = Y_y = Z_z = 0. \tag{A22}$$

The most general solution of (A22) under condition (A21) is of the form

$$X = a_2(y) + a_3(z), \qquad Y = b_1(x) + b_3(z), \qquad Z = c_1(x) + c_2(y);$$

the general conditions (A9), of which (A21) is but a special consequence, cannot be satisfied unless the functions $a_2(y)$, etc. are linear. This leads to the linear functions

$$X = X_0 + a_2 y + a_3 z, \qquad Y = Y_0 + b_1 x + b_3 z, \qquad Z = Z_0 + c_1 x + c_2 y.$$

Even so the coefficients are not free to choose; we must have

$$a_2 + b_1 = 0, \qquad a_3 + c_1 = 0, \qquad b_3 + c_2 = 0.$$

It is now seen that the terms with X_0, Y_0, Z_0, a^*, b^*, c^* in (A13), (A14) and (A15) represent the most general solution to (A22) under (A9). This completes the proof that formulas (A12)–(A15) provide the most general form of the operator $\mathscr{L}$ in order to preserve harmonicity under (A3).

APPENDIX B: EVALUATION OF SOME INTEGRALS

The integrand in (2.17) contains factors

$$\begin{aligned}
X^*\rho\, d\varphi &= -(\sigma_x^* \cos\varphi + \tau_{xz}^* \sin\varphi)\rho\, d\varphi = -\sigma_x^*\, dz + \tau_{xz}^*\, dx \sim -dU_z^* \\
Z^*\rho\, d\varphi &= -(\tau_{xz}^* \cos\varphi + \sigma_z^* \sin\varphi)\rho\, d\varphi = -\tau_{xz}^*\, dz + \sigma_z^*\, dx \sim dU_x^* \\
Y^*\rho\, d\varphi &= -(\tau_{yx}^* \cos\varphi + \tau_{yz}^* \sin\varphi)\rho\, d\varphi = -\tau_{yx}^*\, dz + \tau_{yz}^*\, dx \sim \mu m_3(\beta_x^*\, dz - \beta_z^*\, dx) = -\mu m_3\, d\alpha^*.
\end{aligned} \tag{B1}$$

The analogous terms for X, Y and Z are obtained from the preceding ones by removing the asterisks and by replacing m_j by k_j. We observe that α, U_1, U_2, U_{1z} and U_{2z} and the x-derivatives of these quantities vanish for $\varphi = \pm\pi$. Therefore

$$\int_{\mathscr{L}} dU_z^* = 0, \qquad \int_{\mathscr{L}} dU_x^* = 0, \qquad \int_{\mathscr{L}} d\alpha^* = 0 \tag{B2}$$

which proves (2.18). The limit $I(y)$ is of the form

$$I(y) = \sum_{i,j=1}^{3} a_{ij} m_i k_j \tag{B3}$$

with certain constant coefficients a_{ij}. The asymptotic relations for regular and ordinary fundamental field make it obvious that $a_{31} = a_{32} = a_{23} = a_{13} = 0$. Therefore

$$I(y) = \sum_{i,j=1}^{2} a_{ij} m_i k_j + a_{33} m_3 k_3. \tag{B4}$$

We determine a_{33} first. As a consequence of (B1) we can write

$$\mu a_{33} = \int_{-\pi}^{\pi} (\beta\, d\alpha^* - \beta^*\, d\alpha) = \int_{-\pi}^{\pi} (\beta\, d\alpha_x - \beta_x\, d\alpha)$$

whence

$$\mu a_{33} = \frac{1}{\rho}\int_{-\pi}^{\pi} \beta\, d\alpha. \tag{B5}$$

Along $\mathscr{L}$ we have $2\, d\alpha = -\beta\, d\varphi$, $2\, d\beta = \alpha\, d\varphi$. The first of these relations permits us to rewrite (B5) as follows:

$$\mu a_{33} = -\frac{1}{2\rho}\int_{-\pi}^{\pi} \beta^2\, d\varphi = -\int_{-\pi}^{\pi} \sin^2\varphi/2\, d\varphi = -\pi. \tag{B6}$$

In the absence of terms with k_3, m_3 one may write

$$\begin{aligned}
2\mu I(y) = \int_{-\pi}^{\pi} \{ &-[-U_x + 4(1-\nu)A]\, dU_z^* + [-U_z + 4(1-\nu)B]\, dU_x^* + [-U_x^* + 4(1-\nu)A^*]\, dU_z \\
&- [-U_z^* + 4(1-\nu)B^*]\, dU_x \}.
\end{aligned} \tag{B7}$$

We split

$$2\mu I(y) = J_0 + 4(1-\nu)J_1 \tag{B8}$$

where J_1 is that portion of (B7) where A, A^*, B, B^* appear; J_0 is the remainder. Observing again that U_x, U_z, U_z^* and U_z^* vanish for $\varphi = \pm\pi$ we find

$$J_0 = \int_{-\pi}^{\pi} (U_x\,\mathrm{d}U_z^* + U_z^*\,\mathrm{d}U_x - U_z\,\mathrm{d}U_x^* - U_x^*\,\mathrm{d}U_z) = (U_xU_z^* - U_zU_x^*)\Big|_{-\pi}^{\pi} = 0. \tag{B9}$$

This leaves us with J_1 which we treat by integration by parts as follows:

$$\begin{aligned} J_1 &= \int_{-\pi}^{\pi} (-A\,\mathrm{d}U_z^* + A^*\,\mathrm{d}U_z + B\,\mathrm{d}U_x^* - B^*\,\mathrm{d}U_x) \\ &= \int_{-\pi}^{\pi} (U_z^*\,\mathrm{d}A - U_z\,\mathrm{d}A^* + U_x\,\mathrm{d}B^* - U_x^*\,\mathrm{d}B). \end{aligned} \tag{B10}$$

Let us now consider the special case $k_2 = 0$, $m_1 = 0$. This implies that A, B^*, U_x, U_z^* are even in φ while A^*, B, U_x^*, U_z are odd. Consequently $J_1 = 0$. Altogether we have found that $a_{21} = 0$. The same holds for a_{12}. Returning to (B4) we may now write

$$I(y) = a_{11}m_1k_1 + a_{22}m_2k_2 + a_{33}m_3k_3. \tag{B11}$$

It remains to determine a_{11}, a_{22}. We have

$$\mu a_{11}/(1-\nu) = \int_{-\pi}^{\pi} (U_{1xz}\,\mathrm{d}\alpha - U_{1z}\,\mathrm{d}\alpha_x + U_{1x}\,\mathrm{d}\beta_x - U_{1xx}\,\mathrm{d}\beta) \tag{B12}$$

$$\mu a_{22}/(1-\nu) = \int_{-\pi}^{\pi} (U_{2xz}\,\mathrm{d}\beta - U_{2z}\,\mathrm{d}\beta_x - U_{2x}\,\mathrm{d}\alpha_x + U_{2xx}\,\mathrm{d}\alpha). \tag{B13}$$

Proceeding with (B12) we make use of (2.13) and (2.13a) in particular and find

$$\begin{aligned} \mu a_{11}/(1-\nu) &= \int_{-\pi}^{\pi} [(U_{1xz} - U_{1z}/2\rho)\,\mathrm{d}\alpha - (U_{1xx} + U_{1x}/2\rho)\,\mathrm{d}\beta] \\ &= \frac{1}{4\rho}\int_{-\pi}^{\pi} [(1+\cos\varphi - \sin^2\varphi)\beta^2 - (1+\cos\varphi + \sin^2\varphi)\alpha^2]\,\mathrm{d}\varphi \\ &= -\frac{1}{2}\int_{-\pi}^{\pi} [(1+\cos\varphi)\cos\varphi + \sin^2\varphi]\,\mathrm{d}\varphi = -\pi. \end{aligned} \tag{B14}$$

As for (B13) we observe that

$$\begin{aligned} \mu(a_{11}+a_{22})/(1-\nu) &= 2\int_{-\pi}^{\pi} (\alpha\,\mathrm{d}\beta_x - \alpha_x\,\mathrm{d}\beta) = -\frac{2}{\rho}\int_{-\pi}^{\pi} \alpha\,\mathrm{d}\beta \\ &= -2\cdot\int_{-\pi}^{\pi} \cos^2\varphi/2\,\mathrm{d}\varphi = -2\pi \end{aligned} \tag{B15}$$

whence $a_{22} = a_{11}$. This, together with (B14), (B6) and (B11), leads to

$$I(y) = -\frac{\pi}{\mu}[(1-\nu)(k_1m_1 + k_2m_2) + k_3m_3] \tag{B16}$$

repeated as (2.19) in Section 2.